ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 10

In all problems below, we assume R is a commutative ring.

Problem 1. Assume R is an integral domain. Prove that an ideal $\mathfrak{a}$ is free as an R-module if and only if $\mathfrak{a}$ is principal (i.e. admits one generator).

Problem 2. Assume R is an integral domain, and let M be a free R-module. Use the methods from Lecture 27 to show that any two maximal linearly independent subsets of M have the same cardinality.

Problem 3. Assume that for all finitely generated free modules M over R with rank n we have that every submodule of M is free of rank $\leq n$. Prove that R is a PID.

Problem 4. Prove that $(\mathbb{Q}_{>0}, *)$ is a free $\mathbb{Z}$-module and determine a basis for it.

Problem 5. Fix a module M over R and let $T\colon M \to M$ be an R-linear map. Prove that M is a module over $R[x]$ with scalar multiplication $f(x) \cdot m = f(T)(m)$ for all m in M.

Problem 6. Let $\mathbb{K}$ be a field and $g(x) \in \mathbb{K}[x] \smallsetminus \{0\}$. Show that $\mathbb{K}[x]/(g(x))$ is a $\mathbb{K}$-vector space of dimension $\deg(g)$.

Problem 7. Prove or disprove:

(i) $(\mathbb{Q}, +)$ is a free $\mathbb{Z}$-module;

(ii) $\mathbb{K}(x)$ is a free $\mathbb{K}[x]$-module for any field $\mathbb{K}$.

Problem 8. Assume $M_1, \ldots, M_r$ are R-modules and let $N_i \subset M_i$ be submodules. Show that:

$$\frac{\bigoplus_{i=1}^r M_i}{\bigoplus_{i=1}^r N_i} \simeq \bigoplus_{i=1}^r \frac{M_i}{N_i}.$$

Problem 9. Consider a PID R and let $\mathbf{v} = (a_1, \ldots, a_n)$ be a vector in R^n. Prove that we can extend $\mathbf{v}$ to a basis of the free module R^n if and only if $(a_1, \ldots, a_n) = 1$.

Problem 10. (Modules over non commutative rings) The following exercise provides an example of a non-commutative ring A for which $A^n \simeq A^m$ for all $m, n \in \mathbb{N}$.

Let V be an infinite-dimensional vector space over $\mathbb{R}$ with a countable basis $\{v_n \colon n \in \mathbb{N}\}$. Let $A = \operatorname{End}_{\mathbb{R}} V$. Let $T, T' \in A$ be defined by $T(v_{2n}) = v_n$, $T(v_{2n-1}) = 0$, $T'(v_{2n}) = 0$, $T'(v_{2n-1}) = v_n$ for all $n \geq 1$. Prove that $\{T, T'\}$ is a basis for A as a left A-module. Thus, $A \simeq A^2$. Prove that $A^n \simeq A^m$ for any $m, n \in \mathbb{N}$.

Problem 11. (Free modules with infinite rank)
Let R be a PID and let M be a free R-module. Let F be a submodule of M. We aim to show that F is free.

Let $\{v_i\}_i \in I$ is a basis for M let $M_J = R(v_j : j \in J)$. and $F \neq (0)$.

(i) Consider the sets $F_J = F \cap M_J$ where $J \subset I$, and the set of triples

$$S = \{(F_J, J', w) : J \subseteq I, w\colon J' \to F_J \text{ is a basis for } F_J \text{ indexed by } J' \subset J\}.$$

Show that S is a non-empty set.

(ii) Show that the following relation $\leq$ on S defines a partial order on S:

$$(F_J, J', w) \leq (F_K, K', u) \text{ if } J' \subset K' \text{ and } u_{|_{J'}} = w.$$

In other words, the basis u for F_K is an extension of the basis w for F_J.

(iii) Use Zorn's Lemma on S and show that a maximal element of S has $J = I$, so $F_J = F$. (*Hint:* Use the technique in the proof of Theorem 2 of Lecture 27 (the case where $\operatorname{rank}(M) < \infty$).)

(iv) Conclude that there exists a basis for F indexed by a subset of I (so the rank of a module is well-defined and $\operatorname{rank}(F) \leq \operatorname{rank}(M)$.)

Problem 12. Classify all abelian groups of order 32, 36 and 200.

Problem 13. Prove or disprove:

(i) $(\mathbb{Z}/8\mathbb{Z})^* \simeq (\mathbb{Z}/3\mathbb{Z})^* \times (\mathbb{Z}/3\mathbb{Z})^*$;

(ii) $(\mathbb{Z}/16\mathbb{Z})^* \simeq (\mathbb{Z}/3\mathbb{Z})^* \times (\mathbb{Z}/5\mathbb{Z})^*$;