ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 11

In all problems below, we assume R is a commutative ring and $\mathbb{K}$ is a field.
Two $n \times n$ matrices A and C over $\mathbb{K}$ are *similar* if $A = G^{-1}CG$ for some $G \in \mathrm{GL}_n(\mathbb{K})$

Problem 1. Consider the submodule N of $\mathbb{Z}^3$ generated by $v_1 = (1, 0, 1)$, $v_2 = (2, 3, 1)$, $v_3 = (0, 3, 1)$ and $v_4 = (3, 1, 5)$.

(i) Find a basis for N.
(ii) Find a basis for $\mathbb{Z}^3$ that is compatible with a basis for N (ie., a basis $\{w_1, w_2, w_3\}$ of $\mathbb{Z}^3$ and $a_1, \ldots, a_r \in \mathbb{Z}$ where $r = \mathrm{rank}(N)$, such that $\{a_1w_1, \ldots, a_rw_r\}$ is a basis for N.

Problem 2. Let $R = \mathbb{Q}[X]$ and let N be the submodule of R^3 generated by $v_1 = (2X - 1, X, X^2 + 3)$, $v_2 = (X, X, X^2)$, $v_3 = (X + 1, 2X, 2X^2 - 3)$.

(i) Find a basis for N.
(ii) Find bases for R^3 and N that are compatible (in the sense of Problem 1)

Problem 3. Let R be a PID. Prove that a vector $v = (a_1, a_2, \ldots, a_n)$ in R^n can be completed to a basis of R^n if and only if $\{a_1, a_2, \ldots, a_n\}$ generates the unit ideal.

Problem 4. Find the Smith normal form of the following integral matrices:

$$\begin{pmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 3 & 1 & -4 \\ 2 & -3 & 1 \\ -4 & 6 & -2 \end{pmatrix}.$$

Problem 5. Find the Smith normal forms of the following matrices over $\mathbb{K}[X]$:

$$\begin{pmatrix} X+1 & 2 & -6 \\ 1 & X & -3 \\ 1 & 1 & X-4 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} X-17 & 8 & 12 & -14 \\ -46 & X+22 & 35 & -41 \\ 2 & -1 & X-4 & 4 \\ -4 & 2 & 2 & X-3 \end{pmatrix}.$$

Problem 6. Let R be an integral domain and let $\mathbb{K}$ be the field of fractions of R. Let M be a finitely generated R-module. Let V be the vector space over $\mathbb{K}$ obtained from M, by extension of scalars. Prove that the rank of M is equal to the dimension of V over $\mathbb{K}$.

Problem 7. If R is a PID, and M, N are finitely generated R-modules of rank m and n respectively, prove that $M \oplus N$ is a finitely generated R-module of rank $m + n$. Describe the torsion component of $M \oplus N$.

Problem 8. Let R be a PID, $a \in R \setminus \{0\}$ and $M = R/(a)$. Let p be a prime of R dividing a, and let n be the highest power of p dividing a. Prove that

$$p^{k-1}M/p^kM \simeq \begin{cases} R/(p) & \text{for } k = 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

Problem 9. Let T be the linear operator on $V = \mathbb{C}^2$ whose matrix is $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Is the corresponding $\mathbb{C}[X]$-module cyclic?

Problem 10. Let $R = \mathbb{K}[X, Y]$ be a polynomial ring in two variables over $\mathbb{K}$. Give an example of a module over R, which is finitely generated and torsion free, but not free. Do the same for $R = \mathbb{Z}[X]$.

Problem 11. (Diagonalization of matrices) Prove that $A \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ is diagonalizable over $\mathbb{K}$ (i.e., similar to a diagonal matrix), if and only the minimal polynomial of A has no repeated roots.

Problem 12. Find all possible rational normal forms and Jordan forms of a matrix A whose characteristic polynomial is $(X + 2)^2(X - 5)^3$.

Problem 13. Find all possible rational normal forms and Jordan forms of 8×8 matrices over $\mathbb{C}$ whose minimal polynomial is $X^2(X - 1)^3$.

Problem 14. If N is a $k \times k$ nilpotent matrix such that $N^k = 0$ but $N^{k-1} \neq 0$, prove that N is similar to its transpose. (*Hint:* Prove it for a Jordan block matrix N of size k by finding a permutation matrix P with $N^T = PNP^{-1}$)

Problem 15. Prove that two 2×2 matrices over $\mathbb{K}$ are similar if and only if they have the same minimal polynomial.

Problem 16. Prove that two 3×3 matrices over $\mathbb{K}$ are similar if and only if they have the same minimal and characteristic polynomials.

Problem 17. Let $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be the $\mathbb{Z}$-linear map given by left multiplication by $A \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$. Prove that the image of φ is a subgroup of $\mathbb{Z}^n$ of finite index if and only if $\det(A) \neq 0$. Furthermore, in this case, show that the index equals $|\det(A)|$.

Problem 18. Prove the Cayley-Hamilton Theorem over any commutative ring R: Let $A \in \mathrm{Mat}_{n\times n}(R)$. If $f(X) = \det(X\, I_n - A)$, then $f(A) = 0$.
(*Hint:* Use the identity $\mathrm{Cof}(B)B = B\,\mathrm{Cof}(B) = \det(B)I_n$ for any $n \times n$ matrix B, where $\mathrm{Cof}(B)$ is the cofactor matrix of B).

Problem 19. Let A, B be two $n \times n$ matrices over a field $\mathbb{K}$. Assume that $AB = BA$ and both A and B are diagonalizable (i.e. has a basis of eigenvectors). Show that A and B can be simmultanously diagonalized. (*Hint:* By induction on n. For the inductive step you will need to show that any eigenspace $E_\lambda(A) = \{v \in \mathbb{K}^n : A\,v = \lambda\, v\}$ is invariant under B.)

Problem 20. (Jordan-Chevalley decomposition over algebraically closed fields of characteristic 0)
For any $A \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ we can find $D, N \in_{n\times n} (\mathbb{K})$ with $A = D + N$ where:

- D is diagonalizable (semi-simple),

- N is nilpotent,
- D and N are polynomials in A.

The following steps can be used to prove this result. Consider the characteristic polynomial $\chi_A(x) \in \mathbb{K}[x]$ of A. Factor $\chi_A(x) = \prod_{i=1}^{r}(x-\alpha_i)^{m_i}$ where $\alpha_i \neq \alpha_j$ for $i \neq j$.

(i) If $r = 1$, show that $N = (A - \alpha_i I_n)$ and $D = A - N$ satisfy the requirements.

(ii) If $r \geq 2$, for each $i = 1, \ldots, r$ we can find $P_i(x), Q_i(x) \in \mathbb{K}[x]$ with

$$P_i(x)(x-\alpha_i)^{m_i} + Q_i(x)\prod_{j\neq i}(x-\alpha_j)^{m_j} = 1.$$

Write $B_i(x) := Q_i(x)\prod_{j\neq i}(x-\alpha_j)^{m_j}$.

Show that for any $v \in \mathbb{K}^n$ with $(A - \alpha_i I_n)^{m_i} v = 0$ we get $B_i(A)v = v$. Similarly, show that if $v \in \mathbb{K}^n$ satisfies $(A - \alpha_j I_n)^{m_j} v = 0$ for $j \neq i$, then $B_i(A)v = 0$.

(iii) Conclude that if $r \neq 2$ then $D = \sum_{i=1}^{r} \alpha_i B_i(A)$ is semi-simple and $N = A - D$ is nilpotent.