ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 12

In all problems below, we assume R is a commutative ring and $\mathbb{K}$ is a field.

Problem 1. (Nakayama's Lemma v2)
Consider a finitely generated R-module M and an ideal I of R included in the Jacobson radical of R (see Problem 9, HW 7). If $IM = M$, prove that $M = 0$.

Problem 2. (Nakayama's Lemma v3)
Consider a finitely generated R-module M and an ideal I of R. If $M = IM$, then prove that there exists an element $a \in I$ with $m = am$ for all $m \in M$ (equivalently, $(1-a)M = 0$.).

(*Hint for Problems 1 and 2:* Follow the proof of Nakayama's lemma discussed in Lecture 33.)

Problem 3. (Nakayama's Lemma v3 fails over non-commutative rings)
Consider a non-commutative ring A with no zero divisors and let I be a f.g. proper non-zero idempotent ideal (i.e., $I^2 = I$). Show that Nakayama's Lemma v3 from Problem 2 fails for $M = I$.
(*Note:* Examples of such rings arise in Lie Theory. We can take $\mathfrak{g}$ to be a perfect Lie algebra and let $I = \mathfrak{g}A$ be the augmentation ideal of its enveloping algebra $A = U(\mathfrak{g})$)

Problem 4. (An application of Nakayama's lemma due to Vasconcelos)
The goal of this exercise is to extend Problem 12 HW8 to the non-Noetherian case. More precisely, assume M is a finitely generated R-module and let $f\colon M \to M$ be a surjective R-linear map. We wish to show that f is injective, and hence an isomorphism.

(i) Show that M becomes an $R[x]$ mode via $P(x) \cdot m = P(f)(m)$.

(ii) Show that M is a finitely generated $R[x]$-module with the structure defined in the previous item.

(iii) Show that ideal $I = (x) \subsetneq R[x]$ satisfies $IM = M$.

(iv) Use Problem 2 to find a polynomial $P(x) \in I$ with $m = P(x)m$ for all $m \in M$.

(v) Use item (iv) to conclude that $\mathrm{Ker}(f) = \{0\}$.

Problem 5. Show that bases of vector spaces of any dimension (defined as linearly independent spanning sets) are maximal linearly independent sets.

Problem 6. Consider a $\mathbb{K}$-vector space V and let $W \subset V$ be a subspace. Show that:

$$(V/W)^* = \{\xi \in V^* : \xi_{|W} = 0\}.$$

Problem 7. Consider a collection $\{V_i : i \in I\}$ of finite-dimensional $\mathbb{K}$-vector spaces and let W be a vector space.

(i) Prove that $\mathrm{Hom}_{\mathbb{K}}(\bigoplus_{i\in I} V_i, W) \simeq \prod_{i\in I} \mathrm{Hom}_{\mathbb{K}}(V_i, W)$.

(ii) Prove that $(\bigoplus_{i\in I} V_i)^* \simeq \prod_{i\in I} (V_i)^*$.

(In both items, direct products/sums of vector spaces are defined as the corresponding operations on $\mathbb{K}$-modules.)

Problem 8. Let V and W be two finite-dimensional $\mathbb{K}$ vector spaces, with bases B_V and B_W, respectively. Assume $\dim V = n$ and $\dim W = m$. Consider any linear transformation $f\colon V \to W$ and its corresponding dual map $f^*\colon W^* \to V^*$.

(i) Show that if $A = [f]_{B_V,B_W}$ is the $m \times n$ matrix representative of f with respect to the bases B_V and B_W, then $A^T = [f^*]_{(B_W)^*,(B_V)^*}$ is the matrix representative of f^* with respect to the dual bases $(B_W)^*$ and $(B_V)^*$.

(ii) Show that $W^* \simeq (\mathrm{Im}\, f)^* \oplus \mathrm{Ker}(f^*)$, where we view $(\mathrm{Im}\, f)^* \subset W^*$ by extending a basis of $\mathrm{Im}\, f$ to a basis of W and taking duals.

(iii) Show that the images of f and f^* have the same dimension, and conclude from this that the (column) ranks of A and A^T agree.

Problem 9. Show that given two $\mathbb{K}$-vector spaces V_1 and V_2, the tensor product $V_1 \otimes_{\mathbb{K}} V_2$ defined via universal property (see Lecture 34) is unique up to unique isomorphism. Conclude from this that for any $\mathbb{K}$-vector space V, we have

$$V \otimes_{\mathbb{K}} \mathbb{K} \simeq \mathbb{K} \otimes_{\mathbb{K}} V \simeq V.$$

Problem 10. Consider three $\mathbb{K}$-vector spaces V_1, V_2 and W. Show that:

$$(V_1 \oplus V_2) \otimes_{\mathbb{K}} W \simeq (V_1 \otimes_{\mathbb{K}} W) \oplus (V_2 \otimes_{\mathbb{K}} W).$$

Problem 11. Consider two matrices $X_1 \in \mathrm{Mat}_{m_1\times n_1}(\mathbb{K})$ and $X_2 = \mathrm{Mat}_{m_2\times n_2}(\mathbb{K})$ representing linear transformations $f_1\colon V_1 \to W_1$ and $f_2\colon V_2 \to W_2$ with $\dim V_i = n_i$ and $\dim W_i = m_i$ for $i = 1, 2$. Write down the matrix representing $f_1 \otimes f_2\colon (V_1 \otimes_{\mathbb{K}} V_2) \to (W_1 \otimes_{\mathbb{K}} W_2)$ using the matrices X_1 and X_2.

Problem 12. Apply the construction from Problem 11 to the following matrices:

$$X_1 = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad X_2 = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Problem 12. Consider two square matrices A_1 and A_2 of sizes $n \times n$ and $m \times m$, respectively. Let $\{\lambda_i : i = 1, \dots, n\}$ and $\{\mu_j : j = 1, \dots, m\}$ be the eigenvalues of A_1 and A_2, counted with multiplicity. Show that the eigenvalues of the matrix $A_1 \otimes A_2$ (constructed in Problem 10), counted with multiplicity, are given by $\{\lambda_i \mu_j : i = 1, \dots, n, j = 1, \dots m\}$.

Problem 13. Consider three $\mathbb{K}$-vector spaces U, V, W. Assume V is finite-dimensional. Show that $\mathrm{Hom}_{\mathbb{K}}(V, U) \otimes W \simeq \mathrm{Hom}_{\mathbb{K}}(V, U \otimes W)$ by writing an explicit isomorphism. (*Hint:* When $U = \mathbb{K}$, the statement is the Hom-tensor adjointness theorem)

Problem 14. Consider two finite-dimensional $\mathbb{K}$-vector spaces V and W, each with two bases B_1, B_1' and B_2, B_2', respectively. Describe the change of bases matrix for $V \otimes_{\mathbb{K}} W$ with respect to the bases $B_1 \times B_2$ and $B_1' \times B_2'$ (ordered appropriately).

Problem 15. Consider $\mathbb{K}$-vector spaces U, V, W. Using the universal property of tensor products show that

(i) there exists a unique $\mathbb{K}$-linear isomorphism $\phi : V \otimes_{\mathbb{K}} W \xrightarrow{\simeq} W \otimes_{\mathbb{K}} V$ satisfying $\phi(v \otimes w) = w \otimes v$ for all $v \in V$, $w \in W$;

(ii) there exists a unique $\mathbb{K}$-linear isomorphism $\beta : (U \otimes_{\mathbb{K}} V) \otimes_{\mathbb{K}} W \xrightarrow{\simeq} U \otimes_{\mathbb{K}} (V \otimes_{\mathbb{K}} W)$ satisfying $\beta((u \otimes v) \otimes w) = u \otimes (v \otimes w)$.

Problem 16. Consider two finite-dimensional $\mathbb{K}$-vector spaces V and W, with $\dim V = n$ and $\dim W = m$. Consider the following composition of isomorphisms:

$$\alpha : \mathrm{Hom}_{\mathbb{K}}(V^*, W) \xrightarrow{\simeq} V \otimes_{\mathbb{K}} W \xrightarrow{\phi} W \otimes_{\mathbb{K}} V \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{K}}(W^*, V)$$

where the first and the last maps arise from Hom-tensor adjointness, and ϕ is the map defined in Problem 15 (i). Show that under the identifications $\mathrm{Mat}_{m \times n}(\mathbb{K}) \simeq \mathrm{Hom}_{\mathbb{K}}(V^*, W)$ and $\mathrm{Hom}_{\mathbb{K}}(W^*, V) \simeq \mathrm{Mat}_{n \times m}(\mathbb{K})$, the map α corresponds to taking the transpose of a matrix.

Problem 17. (Rank of tensors)

Consider two $\mathbb{K}$-vector spaces V and W. For any $u \in V \otimes_{\mathbb{K}} W$ we define the *rank* of u as the smallest non-negative integer r for which u admits an expression:

$$u = \sum_{i=1}^{r} v_i \otimes w_i \qquad v_i \in V, w_i \in W. \tag{1}$$

(i) Assume that $\mathrm{rank}(u) = r$ and write u as in (1). Show that the sets $\{v_i : i = 1, \dots, r\}$ and $\{w_i : i = 1, \dots, r\}$ are linearly independent subsets of V and W, respectively.

(ii) Conversely, if $u = \sum_{i=1}^{s} a_i \otimes b_i$ where the sets $\{a_i : i = 1, \dots, s\} \subset V$ and $\{b_i : i = 1, \dots, s\} \subset W$ are linearly independent, then $\mathrm{rank}(u) = s$.

(iii) Assume V and W are finite-dimensional. Hom-tensor adjointness yields isomophisms

$$\varphi_1 : V \otimes_{\mathbb{K}} W \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{K}}(V^*, W), \qquad \varphi_2 : W \otimes_{\mathbb{K}} V \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{K}}(W^*, V).$$

Consider the isomorphism ϕ from Problem 15 item (i) and let $u_1 = \varphi_1(u)$, and $u_2 = (\varphi_2 \circ \phi)(u)$. Show that $\mathrm{rank}(u) = \dim \mathrm{Im}(u_1) = \dim \mathrm{Im}(u_2)$.