ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 12

In all problems below, we assume $\mathbb{K}$ is a field, and U, V and W are $\mathbb{K}$-vector spaces. *Notation:* $T^n(V) = V^{\otimes n}$ is the n-fold tensor power of V; $S^n(V)$ and $\bigwedge^n(V)$ denote the symmetric and exterior n-fold power of V.

Problem 1. Let $k \in \mathbb{Z}_{\geq 1}$ and $v_1, \dots, v_k \in V$. Show that $v_1 \wedge \dots \wedge v_k \in \bigwedge^k V$ is non-zero if, and only if, $\{v_1, \dots, v_k\}$ is linearly independent.

Problem 2. Let $P \in \mathrm{End}_{\mathbb{K}}(V)$ satisfying $P^2 = P$ (i.e. P is a projection). Show that $V \simeq \ker(P) \oplus \mathrm{Im}(P)$.

Problem 3. ($S^n(V)$ as a subspace of $T^n(V)$)

Assume $\mathrm{char}(\mathbb{K}) = 0$ and consider $S\colon T^n(V) \to T^n(V)$ given by $S(\xi) = \dfrac{1}{n!} \displaystyle\sum_{\sigma \in \mathbb{S}_n} \sigma(\xi)$.

(i) Show that $S^2 = S$ and, furthermore,

$$\ker(S) = \langle v_1 \otimes \dots v_n - v_1 \otimes \dots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \dots \otimes v_n : 1 \leq i \leq n-1, v_1, \dots, v_n \in V \rangle.$$

(ii) Conclude that $\mathrm{Im}(S) \simeq S^n(V)$ as $\mathbb{K}$-vector spaces.

Problem 4. ($\bigwedge^n(V)$ as a subspace of $T^n(V)$)

Assume $\mathrm{char}(\mathbb{K}) = 0$ and consider $A\colon T^n(V) \to T^n(V)$ given by $A(\xi) = \dfrac{1}{n!} \displaystyle\sum_{\sigma \in \mathbb{S}_n} \mathrm{sign}(\sigma)\sigma(\xi)$.

(i) Show that $A^2 = A$ and furthermore,

$$\ker(A) = \langle v_1 \otimes \dots v_n + v_1 \otimes \dots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \dots \otimes v_n : 1 \leq i \leq n-1, v_1, \dots, v_n \in V \rangle.$$

(ii) Conclude that $\mathrm{Im}(A) \simeq \bigwedge^n(V)$ as $\mathbb{K}$-vector spaces.

Problem 5. Prove the following isomorphisms of $\mathbb{K}$-vector spaces for all $n \in \mathbb{Z}_{\geq 0}$:

$$S^n(V \oplus W) \simeq \bigoplus_{i=0}^{n} S^i(V) \otimes S^{n-i}(W) \quad \text{and} \quad \bigwedge^n(V \oplus W) \simeq \bigoplus_{i=0}^{n} \bigwedge^i(V) \otimes \bigwedge^{n-i}(W).$$

Problem 6. Let $v_1, \dots, v_k \in V$ be a collection of linearly independent vectors. Let $\omega \in \bigwedge^p(V)$. Show that ω can be written as $\omega = \sum_{i=1}^r v_i \wedge \psi_i$ for some $\psi_1, \dots \psi_r \in \bigwedge^{p-1}(V)$ if, and only if, $v_1 \wedge \dots v_r \wedge \omega = 0 \in \bigwedge^{p+r}(V)$.

Problem 7. Show that the multiplication maps $S^k(V) \times S^\ell(V) \to S^{k+l}(V)$ and $\bigwedge^k(V) \times \bigwedge^\ell(V) \to \bigwedge^{k+\ell}(V)$ defined in Lecture 36 are well-defined and are obtained from the multiplication map $\phi\colon T^k(V) \times T^\ell(V) \to T^{k+\ell}(V)$ composed with the corresponding natural projections to the symmetric and exterior powers of V.

Problem 8. Assume $\operatorname{char}(\mathbb{K}) = 0$ and consider the map $\varphi\colon S^k(V) \times S^\ell(V) \to S^{k+\ell}(V)$ defined on the indecomposable tensors via

$$\varphi(\xi, \eta) = \frac{1}{\binom{k+\ell}{k}} \sum_{\sigma \in G} \sigma(\xi \otimes \eta) \qquad \text{for } G = \mathbb{S}_{k+\ell}/(\mathbb{S}_k \times \mathbb{S}_\ell), \xi \in S^k(V), \eta \in S^\ell(V),$$

where we view $\mathbb{S}_k \times \mathbb{S}_\ell \subset \mathbb{S}_{k+\ell}$ as permutations of $\{1, \ldots, k\}$ and $\{k+1, \ldots, k+\ell\}$.

(i) Check that φ is bilinear, so it yields a unique linear map $\overline{\varphi}\colon S^k(V) \otimes S^\ell(V) \to S^{k+\ell}(V)$.

(ii) Show that $\overline{\varphi}$ defines an associative multiplication map on $S^\bullet(V)$.

(iii) Show that $\overline{\varphi}$ fits into the natural commutative diagram involving the multiplication map on $T^\bullet(V)$, the projection $T^n(V) \to S^n(V)$ and the inclusion $S^n(V) \hookrightarrow T^n(V)$ defined in Problem 3:

$$\begin{array}{ccc} S^k(V) \otimes S^\ell(V) & \xrightarrow{\overline{\varphi}} & S^{k+\ell}(V) \\ \downarrow \quad\quad \downarrow & & \uparrow \\ T^k(V) \otimes T^\ell(V) & \xrightarrow{\text{mult.}} & T^{k+\ell}(V) \end{array}$$

Problem 9. Prove that $V \otimes V \simeq S^2(V) \oplus \bigwedge^2(V)$ by writing the explicit isomorphisms.

Problem 10. Assume V is finite-dimensional. Pick a basis $B = \{v_1, \ldots, v_n\}$ for V.

(i) Show that $\psi_B\colon \bigwedge^n(V) \to \mathbb{K}$ given by $\psi_B(\alpha(v_1 \wedge \ldots \wedge v_n)) = \alpha$ is an isomorphism.

(ii) If B' is another basis for V, and A is the change of bases matrix from B' to B, show that $\psi_B(\xi) = \det(A)\psi_{B'}(\xi)$ for all $\xi \in \bigwedge^n(V)$.

Problem 11. Assume $\operatorname{char}(\mathbb{K}) = 0$. Consider a $\mathbb{K}$-linear map $f\colon V \to W$ and the associated maps $S^n(f)\colon S^n(V) \to S^n(V)$ and $\bigwedge^n(f)\colon \bigwedge^n(V) \to \bigwedge^n(V)$.

(i) Show that these constructions are compatible with compositions, i.e. if $f\colon V \to U$, and $g\colon U \to W$ are $\mathbb{K}$-linear, then

$$S^n(g \circ f) = S^n(g) \circ S^n(f) \qquad \text{and} \qquad \bigwedge^n(g \circ f) = \bigwedge^n(g) \circ \bigwedge^n(f).$$

(ii) Let $n = \dim(V)$ and pick $f \in \operatorname{End}_{\mathbb{K}}(V)$. Show that the following diagram commutes

$$\begin{array}{ccc} \bigwedge^n(V) & \xrightarrow{\bigwedge^n(f)} & \bigwedge^n(V) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathbb{K} & \xrightarrow{\det(f)} & \mathbb{K} \end{array}.$$

Here, the bottom map is multiplication by $\det(f) \in \mathbb{K}$, and the vertical maps are any fixed isomorphism from Problem 10.

(iii) In particular, if $V = U = W$ have dimension n and f and g correspond to two matrices $A, B \in \operatorname{Mat}_{n\times n}(\mathbb{K})$, show that $\det(AB) = \det(A)\det(B)$.

Problem 12. Assume $\mathrm{char}(\mathbb{K}) = 0$. Prove the row expansion formula for determinants of square matrices using $\det(f) = \bigwedge^n(f)\colon \bigwedge^n(\mathbb{K}^n) \to \bigwedge^n(\mathbb{K}^n)$ where $f\colon \mathbb{K}^n \to \mathbb{K}^n$ is multiplication by the corresponding matrix.

Problem 13. (Determinants vs. Permanents of singular matrices)
Assume $\mathrm{char}(\mathbb{K}) = 0$ and consider the endomorphism $f\colon \mathbb{K}^2 \to \mathbb{K}^2$ defined by multiplication by the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

(i) Compute $\bigwedge^2(f)\colon \bigwedge^2(\mathbb{K}^2) \to \bigwedge^2(\mathbb{K}^2)$.

(ii) Compute $S^2(f)\colon S^2(\mathbb{K}^2) \to S^2(\mathbb{K}^2)$.

(iii) Compute $S^3(f)\colon S^3(\mathbb{K}^2) \to S^3(\mathbb{K}^2)$.

Problem 14. Let $X \in \mathrm{GL}_n(\mathbb{K})$. We say X admits a *Gaussian decomposition* if it can be written as a product

$$X = X^- X^0 X^+,$$

where X^0 is a diagonal matrix, X^+ is an upper triangular matrix (i.e. $X^+_{ij} = 0$ for $i > j$) with ones along the diagonal, and X^- is an lower triangular matrix (i.e. $X^-_{ij} = 0$ for $i < j$) with ones along the diagonal.

(i) Show that if X admits a Gaussian decomposition then it is unique. (*Hint:* Prove the uniqueness for diagonal matrices admitting a Gaussian decomposition.)

(ii) Show that a matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{K})$ admits a Gaussian decomposition if, and only if, $a \neq 0$. Compute explicit formulas for X^-, X^0 and X^+.

Bonus Problem: Consider $X \in \mathrm{GL}_n(\mathbb{K})$. The goal of this exercise is to show that X admits a Gaussian decomposition if, and only if, all its principal minors are non-zero.

(i) Show that if X admits a Gaussian decomposition, then all principal minors $\Delta^{1,\ldots,i}_{1,\ldots,i}$ (for $i = 1, \ldots, n$) are non-zero.

(ii) If X admits a Gaussian decomposition, show that $X^0_{11} = X_{11}$ and $X^0_{ii} = \Delta^{1,\ldots,i}_{1,\ldots,i}/\Delta^{1,\ldots,i-1}_{1,\ldots,i-1}$ for all $i = 2, \ldots, n$.

(iii) Furthermore, prove that $X^-_{ji} = \Delta^{1,\ldots,i-1,j}_{1,\ldots,i}/\Delta^{1,\ldots,i}_{1,\ldots,i}$ and $X^+_{ij} = \Delta^{1,\ldots,i}_{1,\ldots,i-1,j}/\Delta^{1,\ldots,i}_{1,\ldots,i}$ for all $i \leq j$.

(iv) Conclude that X admits a Gaussian decomposition if, and only if, all its principal minors are non-zero.