Lecture 3: Order of a group & Basic Isomorphism Theorems

Last Time:
- Defined subgroups ($H < G$), normal subgroups ($H \triangleleft G$) ($gHg^{-1} = H \ \forall g \in G$)
- (normal) Subgroups generated by a set.
- Left cosets $G/H = \{xH : x \in G\}/\sim$ $\quad x \sim y \Leftrightarrow x^{-1}y \in H$
- Right — $H\backslash G = \{Hx : x \in G\}/\sim'$ $\quad x \sim' y \Leftrightarrow xy^{-1} \in H$
- Thm: If $H \triangleleft G$, then G/H is a group under $gH * g'H = gg'H$ & $G \twoheadrightarrow G/H$ is gp hom with $\operatorname{Ker} G = H$.
- Cyclic groups & their classification: G infinite $\simeq \mathbb{Z}$; G finite $\simeq \mathbb{Z}/n\mathbb{Z}$ ($n = |G|$)
- Hamiltonian groups (Example: Quaternions Q_8 via gens & relations)

TODAY Discuss 3 Isomorphisms in Group Theory.

§1. More on cosets & First counting Lemma:

Def $|G| = \#$ elements in G is called the order of G.

Eg: $|S_n| = n!$ $\qquad |\mathbb{Z}/n\mathbb{Z}| = n$.

- If $H < G$, then G breaks into a disjoint union of left cosets

$$\boxed{G = \bigsqcup_{\alpha \in A} g_\alpha H}$$

A = choice of representatives of G/H

In particular, A is in bijection with G/H. This gives us our first counting lemma.

Lemma: Assume G is finite. Then $|G| = |H|\,|G/H|$

Pf/ For each g $\varphi_g : H \longrightarrow gH$, $h \longmapsto gh$ is a bijection.

Corollary: $|H|$ divides $|G|$

Remark: $|G/H|$ is usually denoted by $(G:H)$ = index of H in G

It is possible for both G & H to be infinite & yet $(G:H)<\infty$.

Example: $G=\mathbb{Z}$, $H=5\mathbb{Z}$ infinite but $(G:H)=5<\infty$

Def: If $(G:H)<\infty$ we say H is a finite index subgroup.

- Any $g\in G$ generated a subgroup $\langle g\rangle$. So we define:

Def The order of an element g of G is the order of $H=\langle g\rangle$.

Obs.: If $|H|=n<\infty$, then $g^n=e$ $\left(H=\{1,g,\ldots,g^{n-1}\}\simeq \mathbb{Z}/n\mathbb{Z}\right)$.
$g^\ell \longmapsto \bar{\ell}$

Corollary: $\text{Order}(g) \mid |G|$ whenever G is finite.

Def: Exponent of G $(\exp(G))$ = generator of $\{k\in\mathbb{Z} : g^k=e \ \forall g\in G\}\cap\mathbb{Z}_{\geq 0}$

Obs: • If $|G|<\infty$, then $\exp(G)>0$ $\quad(g^{|G|}=e \ \forall g\in G)$

Converse is false: $G=\prod(\mathbb{Z}/2\mathbb{Z})=\{(a_1,a_2,\ldots,) \ a_i=0,1 \ \forall i\}$
with term-by-term multiplication has $\exp(G)=2$

Prop:
- $\exp(G)=1 \implies G=\{e\}$
- ——— $=2 \implies G$ is abelian (Exercise)
- ——— $=3$ need not be abelian

[Ex: Heisenberg gp / $\mathbb{F}_3$: $H_3=\left\{\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix} \middle| a,b,c\in\mathbb{Z}/3\mathbb{Z}\right\}<Gl_3(\mathbb{F}_3)$]

Burnside Problem (1902). Find all $(n,m)\in\mathbb{Z}^2_{>0}$ such that if G is a group with m generators (minimal #) & $\exp(G)=n$, then $|G|<\infty$.

Status: Known cases: $(1,n)$, $(2,n)$ many ... Still OPEN! [$(2,5)$???]
$(m,3)$, $(m,4)$, $(m,6)$ many. (see Wikipedia)

§2 First Isomorphism Theorem:

Theorem 1: Let G, G' be two groups and $\varphi : G \longrightarrow G'$ be a group homomorphism. Write $K = \operatorname{Ker}(\varphi) \lhd G$ & $H' = \operatorname{Im}(\varphi) < G'$.

Then we have a commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G' \\ G/K \arrow[r, "\overline{\varphi}"'] \arrow[ur, "\exists \psi"] & H' \arrow[u, "i"'] \end{tikzcd}$$

$(\overline{\varphi}(gK) = \varphi(g))$

Here: π = natural projection & i is a natural inclusion.

Moreover, $\overline{\varphi}$ is an isomorphism.

Proof: Define $\psi : G/K \longrightarrow G'$ by $\psi(gK) = \varphi(g)$

Claim 1: ψ is well-defined $(g_1 K = g_2 K \overset{?}{\Longrightarrow} \varphi(g_1) \overset{?}{=} \varphi(g_2))$

Pf/ $g_1 K = g_2 K \Longleftrightarrow g_2^{-1} g_1 \in K$, so $\varphi(g_2^{-1} g_1) = \varphi(g_2)^{-1} \varphi(g_1) = e'$

Thus: $\varphi(g_1) = \varphi(g_2)$ ✓

Claim 2: ψ is a group homomorphism:

Pf/ $\psi(g_1 K\, g_2 K) = \psi(g_1 g_2 K) = \varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) = \psi(g_1 K)\, \psi(g_2 K)$

Claim 3: ψ is injective

Pf/ $\psi(gK) = e' \Longleftrightarrow \varphi(g) = e' \Longleftrightarrow g \in K \Longleftrightarrow gK = K$.

Claim 4: $\overline{\varphi} = \psi$ with range restricted to $H' = \operatorname{Im}(\varphi)$

By definition $\overline{\varphi}$ is surjective & injective, so it is a bijection.

Exercise: Bijective group homomorphisms are isomorphisms (HW1). $\square$

The other two isomorphism theorems will follow from this one.

$\boxed{\underline{\text{First Iso Thm}}: \quad \frac{G}{\ker\varphi} \xrightarrow{\sim} \operatorname{Im}\varphi}$

§3 Second Isomorphism Thm:

__Thm 2:__ Let G be a group and $N \lhd G$ a normal subgp. Then

(i) The assignment $H \longrightarrow H/N$ is a bijection between

$$\left\{\begin{matrix}\text{Subgroups of}\\ G \text{ containing } N\end{matrix}\right\} \longleftrightarrow \{\text{Subgroups of } G/N\}$$

(ii) Let $H < G$ be a subgroup containing N. Then

H is normal __if and only if__ H/N is normal in G/N

Furthermore, we have $G/H \xrightarrow{\sim} \frac{G/N}{H/N}$ $\quad gH \longmapsto gH/N$

__Proof of (i)__ Let $\pi: G \longrightarrow G/N$ be the natural surjection. & pick $H<G$

__Claim 1__ If $N \subseteq H$, then $\pi(H) = \{hN : h \in H\} < G/N$

Pf/ • $e_{G/N}$ = identity of G/N $= eN \in \pi(H)$ ✓

• $(h_1N)(h_2N) = h_1h_2N$ $\forall h_1,h_2 \in H$, so law of composition holds for $\pi(H)$

• $(hN)^{-1} = h^{-1}N$ $\forall h \in H$, so $\pi(H)$ is closed under inverses. □

For the converse, pick $\overline{H} < G/N$ a subgroup, let $H = \pi^{-1}(\overline{H})$.

__Claim 2__: $H<G$ is a subgroup of G containing N & $\pi(H) = \overline{H}$.

Pf/ $N = \pi^{-1}(\{e_{G/N}\}) = \ker\pi \subset \pi^{-1}(\overline{H}) \overset{\text{def}}{=} H$.

By def: $H = \{g \in G : \pi(g) \in \overline{H}\}$ __Want to show: $H<G$__

• $e \in H$ is clear since $e \in N \subset H$ ✓

• $g_1, g_2 \in H \Rightarrow \pi(g_1g_2) = \pi(g_1)\pi(g_2) \in \overline{H}\cdot\overline{H} = \overline{H}$.

so $g_1g_2 \in H$. ✓

• $g \in H \Rightarrow \pi(g^{-1}) = \pi(g)^{-1} \in \overline{H}^{-1} = \overline{H} \Rightarrow g^{-1} \in H$ ✓

Finally: $\pi(H) = \overline{H}$ by the surjectivity of π □

Proof of (ii): Fix H subgroup of G with $N \subset H$ & set $\boxed{\overline{H} := \pi(H) = H/N}$

$$H \lhd G \iff ghg^{-1} \in H \ \forall g \in G\ \forall h \in H \iff ghg^{-1}N \in \overline{H} \ \forall g \in G\ \forall h \in H$$

$$\iff (gN)(hN)(g^{-1}N) \in \overline{H} \quad \forall g \in G \quad \forall h \in H \iff \overline{H} \lhd G/N$$

. To finish, assume $N \subset H$ & $N \lhd G$, $H \lhd G$. Then

$$\begin{tikzcd} G \arrow[r, two heads, "\pi_1"] \arrow[rr, bend right, "\psi = \pi_1 \circ \pi_2"'] & G/N \arrow[r, two heads, "\pi_2"] & G/N \big/ H/N \end{tikzcd}$$

- ψ is composition of gp homomorphisms, so it is also a gp hom.
- ψ is surjective (comp of surjections)
- $\underline{\operatorname{Ker} \psi = H}$: $\psi(g) = e \iff \pi_1(g) \in H/N \iff gN \in H/N$

$$\iff gN = hN \text{ for some } h \iff g \in hN \subseteq H.$$

Now, by 1^{st} Isomorphism Thm: $\quad G/H \underset{\psi}{\simeq} G/N \big/ H/N$. $\square$

. Asides from the proofs of both (i) & (ii):

Prop 1: For any group homomorphism $\varphi: G_1 \to G_2$, if $H_1 < G_1$ is a subgp then $\varphi(H_1) < G_2$ is a subgroup.

Prop 2: For any group homomorphism $\varphi: G_1 \to G_2$ & $N_2 \lhd G_2$, then $\varphi^{-1}(N_2) \lhd G_1$.

§4. Third Isomorphism Thm:

Thm 3: Let G be a group, $H < G$ a subgroup & $N \lhd G$. Then:

(i) $H \cap N \subset H$ is a normal subgroup

(ii) $HN := \{ hx : h \in H, x \in N \} \subset G$. Then $HN = NH$ & HN is a subgroup of G.

(iii) $N \subset HN$ is a normal subgroup.

[(iv) $$\begin{aligned} H/H\cap N &\longrightarrow HN/N \\ h(H \cap N) &\longmapsto hN \end{aligned}$$ is an isomorphism]

$\boxed{\text{Cartoon encoding (i) — (iii)}}$

$H < G$, $N \lhd G$ $\rightsquigarrow$

$$\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash, "\vee"] & \\
 & HN = NH \arrow[dl, dash, "<"'] \arrow[dr, dash, "\rhd"] & \\
H \arrow[dr, dash, "\rhd"'] & & N \arrow[dl, dash, "<"] \\
 & H \cap N &
\end{tikzcd}$$

Proof of (i): Want to show $H \cap N \lhd H$. Pick $h \in H$ & $x \in H \cap N$.

Then: $\left.\begin{array}{l} \cdot\ hxh^{-1} \in H \text{ because } H < G \\ \cdot\ hxh^{-1} \in N \text{ —— } N \lhd G \end{array}\right\} \Rightarrow hxh^{-1} \in H \cap N$ $\square$.

Proof of (ii): We first show $HN \subseteq NH$. Pick $hx \in HN$ ($h \in H$, $x \in N$)

Claim 1 $hx \in NH$ $\left(\text{Pf/ } hx = \underbrace{hxh^{-1}}_{\in N \lhd G} h \in NH \checkmark \right)$

Proof of $NH \subset HN$ is similar.

Claim 2: HN is a subgroup of G

Pf/ . $e = e \cdot e \in HN$ ✓

- $(h_1 x_1)(h_2 x_2) = h_1 h_2 \underbrace{h_2^{-1} x_1 h_2}_{\in N\ (N \lhd G)} x_2 \in HN$

- $(hx)^{-1} = x^{-1} h^{-1} \in NH = HN$ by Claim 1 for all $h \in H$, $x \in N$. $\square$

Proof of (iii): $N \trianglelefteq G$ & $N < HN \subset G \Rightarrow N \trianglelefteq HN$.

Proof of (iv): Consider the composition of group homomorphisms

$$\begin{tikzcd} H \arrow[r, hook, "i", "\text{inclusion}"'] \arrow[rr, bend left, "\varphi = \pi \circ i"] & HN \arrow[r, two heads, "\pi", "\text{projection}"'] & HN/N \end{tikzcd}$$

: φ is a group homomorphism

Claim 1: φ is surjective

Pf/ $hxN = hN$ for $h \in H$, $x \in N$
But $hN = i(h)$ $\Big\}$ $\varphi(h) = hxN$.

Claim 2 $\operatorname{Ker} \varphi = H \cap N$

Pf/ $h \in \operatorname{Ker} \varphi$ $(h \in H)$ $\Leftrightarrow$ $\varphi(h) = \bar{e}$ (= identity of HN/N) (& $h \in H$)
$\Leftrightarrow$ $hN = N$ $(h \in H)$ $\Leftrightarrow$ $h \in H \cap N$

Then, by First Isomorphism Thm, we get $\dfrac{H}{H \cap N} \simeq \dfrac{HN}{N}$