Lecture 5: Group Actions on Sets

So far: (1) Defined useful terms from Group Theory:

- Group, subgroup, subgroup generated by a subset, order & exponent
- Normal subgroup, normal subgp ————
- Left / Right cosets (G/H & $H\backslash G$), Quotient groups
- Group homomorphisms / Isomorphisms, Kernel & Image of gp hom.
- Free group, Generators & relations; Examples ($\mathrm{Free}(A)$, S_n, D_n)

(2) Main Results: 3 Isomorphism Thms, Classification of cyclic gps.

TODAY: Groups acting on sets

§1. Group actions:

Def: Let G be any group and let X be a set. A (left) action of G on X is a set map

$$G \times X \xrightarrow{\alpha} X \qquad (g, x) \longmapsto \alpha(g,x) =: g \cdot x$$

satisfying

NOTATION $G \circlearrowright X$

(i) $e \cdot x = x \quad \forall x \in X$

(ii) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \quad \forall g_1, g_2 \in G$ and $x \in X$.

[For a right action we replace (ii) by (ii') $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$]

$X \circlearrowleft G$

Observation: If $G \circlearrowright X$, then each $g \in G$ defines a set map:

$$\tau(g) := \alpha(g, -) : X \longrightarrow X \qquad x \longmapsto g \cdot x$$

It satisfies:

(i) $\tau(e) = \mathrm{Id}_X$

(ii) $\tau(g_1 g_2)(x) = (g_1 g_2) x = g_1 \cdot (g_2 \cdot x) \quad \tau(g_1)(\tau(g_2)(x)) = \tau(g_1) \circ \tau(g_2)$

(iii) $\tau(g_1^{-1}) \circ \tau(g_1) = \tau(g_1^{-1} g_1) = \tau(e) = \mathrm{Id}_X$ by (i)

Conclusion: $\tau : G \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(X) := \{ f : X \to X \text{ bijection} \}$ is a gp homomorphism.

$= \text{Symmetric gp on } X$

<u>Example</u> $G = GL_n(\mathbb{R}) \circlearrowright X = \mathbb{R}^n$ by

$$\begin{aligned} G \times X &\xrightarrow{\alpha} X \qquad \text{matrix multiplication.} \\ (A, \underline{x}) &\longmapsto A\underline{x} \end{aligned}$$

Want to highlight that for all $A \in GL_n(\mathbb{R})$ the resulting map $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$ is not just a set bijection, but it <u>preserves</u> the <u>vector space structure</u> $(A(\alpha v_1 + \beta v_2) = \alpha A(v_1) + \beta A(v_2)$ $\forall \alpha, \beta \in \mathbb{R}$ $\forall v_1, v_2 \in \mathbb{R}^n)$

Thus, $GL_n(\mathbb{R}) = \mathrm{Aut}_{\mathbb{R}\text{-v.s.}}(\mathbb{R}^n)$.

§2 Orbits, Stabilizers & Fixed Points

Fix $G \circlearrowright X$.

<u>Def</u>: The <u>orbit</u> of an element $x \in X$ is the following subset of X

$$\boxed{G \cdot x} := \{ g \cdot x \mid x \in G \} \subseteq X$$

<u>Def</u> The <u>stabilizer</u> of an element $x \in X$ is the following subgroup of G

$$\boxed{\mathrm{Stab}_G(x)} := \{ g \in G \mid g \cdot x = x \} \leq G$$

<u>Obs</u>: Stab_G need <u>not</u> be a normal subgroup (Example on page 5)

<u>Def</u>: The <u>fixed point set</u> of an element $g \in G$ is the following subset of X:

$$\boxed{X^g} := \{ x \in X \mid g \cdot x = x \} \subseteq X$$

<u>Example</u>: $D_n \hookrightarrow GL_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{R}^2)$

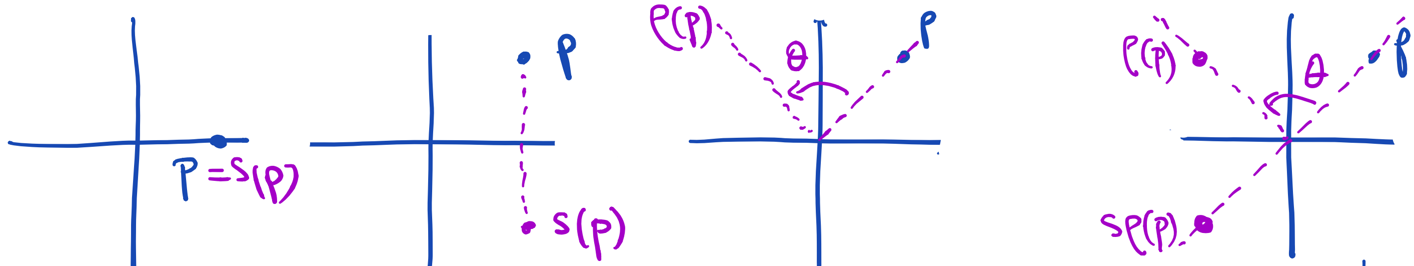
$$s \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(reflection about x-axis)}$$

$$\rho \longmapsto \begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix} \quad (\text{rotation of angle } \theta = \tfrac{2\pi}{n})$$

- This is a group homomorphism, so it defines an action $D_n \circlearrowright \mathbb{R}^2$.

Also on $X = \mathbb{R}^2 \setminus \{ \binom{0}{0} \}$. [$D_n$ fixes $\binom{0}{0}$, so we can ignore it]

• Let us compute the orbit of a pt $p \in X$.



$s\rho = \rho^{-1}s$

In particular, $\boxed{|\{p, \rho(p), \dots, \rho^{n-1}(p)\}| = n}$ $(*)$

$D_n \cdot p = \{p, \rho(p), \rho^2(p), \dots\dots, \rho^{n-1}(p), s_{(p)}, s\rho_{(p)}, \dots, s\rho^{n-1}_{(p)}\}$

$\supseteq \{p, \rho(p), \dots\dots \rho^{n-1}(p)\}$

Claim: $|D_n \cdot p| = 2n \iff s(p) \notin \{p, \rho(p), \dots, \rho^{n-1}(p)\}$

Pf/ $(\Rightarrow)$ If $s(p) = \rho^r(p)$ for some $r = 0, \dots, n-1$, we have a repeated element Contr!

$(\Leftarrow)$ If $|D_n \cdot p| < 2n$, then there is a repeated element x.

Now $x = s\rho^k(p)$ or $\rho^j(p)$ for some k, j

BUT $s\rho^k(p) \neq s\rho^j(p)$ & $\rho^k(p) \neq \rho^j(p)$ by $(*)$. So the only option is $x = s\rho^k(p) = \rho^j(p)$ for some k, j

BUT $s\rho^k = \rho^{-k}s = \rho^{n-k}s \Rightarrow \rho^{n-k}s(p) = \rho^j(p)$

$s(p) = \rho^{k+j}(p)$ Contr!

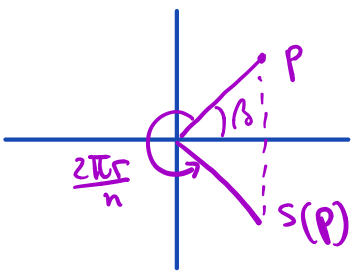
□

Now: $s(p) = \rho^r(p)$ for some $r = 0, \dots, n-1$ means

Reflecting p about x-axis $=$ rotation by $\frac{2\pi r}{n}$.

So $p = Re^{-\beta i} = Re^{(\beta + \frac{2\pi r}{n})i}$ Then $\beta + \frac{2\pi r}{n} \equiv -\beta \ (2\pi)$

$\beta \equiv -\frac{\pi r}{n} \ (\pi)$



So $p = R \begin{bmatrix} \cos \frac{\pi r}{n} \\ -\sin \frac{\pi r}{n} \end{bmatrix}$ & $|D_n \cdot p| = n$.

Q: $\mathrm{Stab}_{D_n}(p) = ?$

• $s\rho^j(p) = \rho^{-j}s(p) \underset{\text{def } p}{=} \rho^{r-j}(p) \rightsquigarrow \frac{\pi r}{n} + \frac{2\pi(r-j)}{n} \equiv \frac{\pi r}{n} \ (2\pi)$

• $\rho^j(p) \neq p \quad \forall j = 1, \dots, r$ $\quad \iff n \mid r-j \rightsquigarrow \boxed{j = r}$

Conclusion: $\mathrm{Stab}_{D_n}(p) = \{s\rho^r, e\} = \langle s\rho^r \rangle$

□

Proposition: Let $G \circlearrowright X$

(1) For every $x \in X$ we have a (set) bijection
$$G/\mathrm{Stab}_G(x) \longrightarrow G\cdot x$$

(2) For every $\sigma \in G$ and $x \in X$, we have an isomorphism of groups
$$\begin{aligned} \mathrm{Conj}_\sigma|_{\mathrm{Stab}_G(x)} : \mathrm{Stab}_G(x) &\longrightarrow \mathrm{Stab}_G(\sigma\cdot x) \\ g &\longmapsto \sigma g \sigma^{-1} \end{aligned}$$
(conjugation by σ)

Proof: (1) Define $f := \alpha(-,x) : G \longrightarrow G\cdot x$, $g \longmapsto g\cdot x$. Surjective by definition.

But $g\cdot x = h\cdot x \iff h^{-1}g\cdot x = x \iff h^{-1}g \in \mathrm{Stab}_G(x)$

So this map factors through $G/\mathrm{Stab}_G(x)$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G\cdot x \\ G/\mathrm{Stab}_G(x) \arrow[ur, "\bar{f}", "\sim"'] & \end{tikzcd}$$

$\bar{f}$ is a bijection

$\bar{f}(g\,\mathrm{Stab}_G) = g\cdot x = f(g)$.

(2) $g \in \mathrm{Stab}_G(x) \iff g\cdot x = x \iff (\sigma g \sigma^{-1})(\sigma x) = \sigma x$
$\iff \sigma g \sigma^{-1} \in \mathrm{Stab}_G(\sigma x)$

Since Conj_σ is an automorphism of G, it induces an isomorphism between $\mathrm{Stab}_G(x)$ & $\mathrm{Stab}_G(\sigma x)$. $\square$

§3 Properties of group actions:

There are 3 properties of group actions that are very useful in Diff'l & Alg. Geometry, Rep-n Theory, Topology ("orbifolds", quotient spaces), etc.

(1) **Free** We say a G-action on X is free if $g\cdot x = x \implies g = e$ for some x

Equivalently
- $\mathrm{Stab}_G(x) = \{e\}$ $\forall x \in X$
- If G is finite, then all orbits have the same size $= |G|$

(2) Transitive: We say a G-action on X is transitive if $\forall x, y \in X$, $\exists g \in G$ such that $g \cdot x = y$

Equivalently: $G \cdot x = X$ for all $x \in X$. (only one orbit)

(3) Faithful: We say a G-action on X is faithful if $G \xrightarrow{\tilde{}} \mathrm{Aut}_{Set}(X)$ is injective (G is faithfully represented in $\mathrm{Aut}_{Set}(X)$)

Equivalently: $g \cdot x = x \ \ \underline{\forall} x \in X \implies g = e$

Obs: Free $\implies$ Faithful but Faithful $\not\Rightarrow$ Free

Examples: ① $D_n \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$
- Faithful ✓
- Free ✗ ($\exists$ orbits of size $n \neq |D_n|$)
- Transitive ✗ (there are many orbits!)

② $S_n \curvearrowright \{1, 2, \ldots, n\}$
- Faithful ✓ $S_n = \mathrm{Aut}(\{1, \ldots, n\})$
- Free ✗ (S_{n-1} fixes $\{n\}$)
- Transitive ✓ (induct on n)

$[S_{n-1} = \mathrm{Stab}_{S_n}(n) \ntrianglelefteq S_n]$

③ $G \curvearrowright G/H$ by $g(g'H) = gg'H$
Faithful ?
Free ?
Transitive ? (Exercise)

§4 Counting Lemmas

Def: $x \sim_G x'$ in X iff $\exists g \in G \ \ g \cdot x = x'$

Claim: This defines an equivalence relation.

$$\boxed{G\backslash X := X/\sim_G \quad \text{equiv classes} = \text{orbits}}$$

Easy Observation: $X = \bigsqcup_{\alpha \in G\backslash X} G \cdot x_\alpha \implies |X| = \sum_{\alpha \in G\backslash X} |G \cdot x_\alpha|$

(Here $x_\alpha \in X$ is a choice of an element from the G-orbit labeled by $\alpha \in X/G$)

Recall: (Prop) $G/\mathrm{Stab}_G(x) \xrightarrow{\text{bij}} G\cdot x$ & $\mathrm{Stab}_G(x) \xrightarrow{\text{Conj}\,\sigma} \mathrm{Stab}_G(\sigma\cdot x)$

Corollary: (a) $|G| = |G\cdot x|\,|\mathrm{Stab}_G x| \quad \forall x\in X$

(b) $|X| = \sum_{\alpha\in G\backslash X} |G| / |\mathrm{Stab}_G(x_\alpha)|$

Burnside' Lemma: # of orbits = average # of fixed pts

More precisely: $|G\backslash X| = \frac{1}{|G|}\sum_{g\in G}|X^g|$

Proof: Write $F := \{(g,x)\in G\times X \mid g\cdot x = x\} \subseteq G\times X$

$$\begin{array}{ccc} & G\times X & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ G & & X \end{array}$$

(incidence corresp.)

Claim 1: $|F| = \sum_{g\in G}|X^g|$

Pf/ Sum over 1^{st} Component (Fix g: $(g,x)\in F \iff x\in X^g$)

Claim 2: $|F| = \sum_{x\in X}|\mathrm{Stab}_G(x)|$

Pf/ Sum over 2^{nd} component (Fix $\underline{x}$: $(g,x)\in F \iff g\in\mathrm{Stab}_G(x)$)

Combining the two claims:

$X = \bigsqcup_{\alpha\in G\backslash X} G\cdot x_\alpha$

$$|F| = \underbrace{\sum_{g\in G}|X^g|} = \sum_{x\in X}|\mathrm{Stab}_G(x)| = \sum_{\alpha\in G\backslash X}\ \underbrace{\sum_{y\in\alpha}|\mathrm{Stab}_G(y)|}_{=|\mathrm{Stab}_G(x_\alpha)|\ (y=\sigma\cdot x_\alpha)}$$

$$= \sum_{\alpha\in G\backslash X}\underbrace{|\mathrm{Stab}_G(x_\alpha)|\,|G\cdot x_\alpha|}_{=|G|} = \underbrace{|G|\,|G\backslash X|}$$

So $\frac{1}{|G|}\sum_{g\in G}|X^g| = |G\backslash X|$ ✓

□

§5 Application

Prop: Given any prime $p \in \mathbb{Z}_{\geq 2}$ & $m \in \mathbb{Z}_{\geq 1}$, we have $\binom{p^r m}{p^r} \equiv m \pmod{p}$

Proof: (1) Induct on r

(2) Use group actions! Take $G = \mathbb{Z}/p^r\mathbb{Z}$, $X = \{x_1, \ldots, x_m\}$ any set with m elements.

- E = set of all p^r element subsets of $G \times X$. $\rightsquigarrow$ $|E| = \binom{p^r m}{p^r}$
- $G \circlearrowright G \times X$ by $\sigma(g, x) = (\sigma \cdot g, x)$

  So $G \circlearrowright E$ by $\sigma \{e_1, \ldots, e_{p^r}\} = \{\sigma(e_1), \sigma(e_2), \ldots, \sigma(e_{p^r})\} \in E$ ✓

  (axioms for left action are satisfied)

- Write E as a disjoint union of orbits:

  For each $\vartheta \in G\backslash E$ (orbit representative) ϑ represented by $S_\vartheta \in E$.

  $|G.S_\vartheta| = \dfrac{|G|}{|\mathrm{Stab}_G(S_\vartheta)|}$ $\mid$ $|G| = p^r$ so $|G.S_\vartheta| = 1$ or $p \mid |G.S_\vartheta|$

Conclude: $\boxed{|E| \equiv \#(\text{of orbits with exactly one element}) \pmod{p}}$

Let's count how many such orbits we have: 2^{nd} entry of each member of the orbit is fixed!

$\Rightarrow$ Orbits are $\{(g, x_1) : g \in G\}$, $\{(g, x_2) : g \in G\}$, …, $\{(g, x_m) : g \in G\}$

$\Rightarrow$ m of them!

We get $\binom{p^r m}{m} = |E| \equiv m \pmod{p}$ $\square$

Obs: This will be used in one of the Sylow Theorems.