Lecture 6: Group Actions on Sets II

Recall G a group, X a set, we define a left action of G on X as a map

$$\begin{aligned} G\times X &\longrightarrow X \\ (g,x) &\longmapsto g\cdot x \end{aligned} \quad \text{satisfying} \quad \begin{aligned} &(i)\ e\cdot x = x \quad \forall x\in X \\ &(ii)\ (g_1 g_2)\cdot x = g_1\cdot(g_2\cdot x) \quad \forall g_1,g_2\in G\ \forall x\in X \end{aligned}$$

Equivalently:
$$\begin{aligned} G &\longrightarrow \mathrm{Aut}_{\mathrm{Set}}(X) \\ g &\longmapsto (x \longmapsto g\cdot x) \end{aligned}$$
is a group homomorphism. **Write** $G \circlearrowright X$

Key subsets: orbits, stabilizers & fixed pt sets.

- Orbit of $x\in X$ is $\boxed{G\cdot x} = \{g\cdot x : g\in G\} \subseteq X$
- Stabilizer of $x\in X$ is $\boxed{\mathrm{Stab}_G x} = \{g\in G \mid g\cdot x = x\} < G$ (subgroup, but generally not normal)
- Fix point set for $g\in G$ is $\boxed{X^g} = \{x\in X \mid g\cdot x = x\} \subseteq X$

- Equivalence Relation on X:

$$\begin{aligned} x \sim_G y &\iff \exists g\in G : g\cdot x = y \\ &\iff x \,\&\, y \text{ in same } G\text{-orbit} \end{aligned}$$

$$\rightsquigarrow \boxed{G\backslash X := X/\sim_G} \quad \text{partition of } X \text{ into Equivalence Classes}$$

- Write this partition as $X = \bigsqcup_{\alpha\in G\backslash X} G\cdot x_\alpha \implies$ (*) $\boxed{|X| = \sum_{\alpha\in G\backslash X} |G\cdot x_\alpha|}$

(Here $x_\alpha\in X$ is a choice of an element from the G-orbit labeled by $\alpha\in X/G$)

§1 Counting Lemmas:

Our first objective is to count the orbits of $G \circlearrowright X$

Recall: (Prop) $G/\mathrm{Stab}_G(x) \xrightarrow{\text{bij}} G\cdot x$ & $\mathrm{Stab}_G(x) \xrightarrow{\mathrm{Conj}_\sigma} \mathrm{Stab}_G(\sigma\cdot x)$

Corollary: (a) $|G| = |G\cdot x|\,|\mathrm{Stab}_G\, x| \quad \forall x \in X$

(b) $$|X| = \sum_{\alpha \in G\backslash X} \frac{|G|}{|\mathrm{Stab}_G(x_\alpha)|}$$

Example: $S_n \circlearrowright \{1,\ldots n\}$ Only 1 orbit $= X$.

$\mathrm{Stab}_{S_n}(n) \simeq S_{n-1} \hookrightarrow S_n \Rightarrow n = |X| = \dfrac{|G|}{|\mathrm{Stab}_G(n)|} = \dfrac{n!}{(n-1)!}$ □

Our first main result is the following:

Burnside' Lemma (Frobenius, 1887) # of orbits = average # of fixed pts

More precisely: $$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G} |X^g|$$

Proof: Write $F := \{(g,x) \in G\times X \mid g\cdot x = x\} \subseteq G\times X$

$$\begin{tikzcd} & G\times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & X \end{tikzcd}$$

(incidence corresp.)

Claim 1: $$|F| = \sum_{g\in G} |X^g|$$

Pf/ Sum over 1st component (Fix g: $(g,x) \in F \Leftrightarrow x \in X^g$)

Claim 2: $$|F| = \sum_{x\in X} |\mathrm{Stab}_G(x)|$$

Pf/ Sum over 2nd component (Fix $\underline{x}$: $(g,x)\in F \Leftrightarrow g \in \mathrm{Stab}_G(x)$)

Combining the two claims:

$$|F| = \underbrace{\sum_{g\in G}}\, |X^g| = \sum_{x\in X} |\mathrm{Stab}_G(x)| \overset{X = \bigsqcup_{\alpha\in G\backslash X} G\cdot x_\alpha}{=} \sum_{\alpha\in G\backslash X}\ \sum_{y\in\alpha} \underbrace{|\mathrm{Stab}_G(y)|}_{= |\mathrm{Stab}_G(x_\alpha)| \ (y = \sigma\cdot x_\alpha)}$$

$$= \sum_{\alpha\in G\backslash X} \underbrace{|\mathrm{Stab}_G(x_\alpha)|\,|G\cdot x_\alpha|}_{=|G|} = \underbrace{|G|\,|G\backslash X|}$$

So $$\frac{1}{|G|}\sum_{g\in G}|X^g| = |G\backslash X| \quad ✓$$

□

Example $S_n \circlearrowright \{1,2,\dots n\} = X$ $\quad |X^\sigma| = \#$ 1-cycles in σ

$\sigma \in S_n$ $\quad$ (Eg $n=5$: $X^{(123)(4)(5)} = \{4,5\}$)

Induces $H = \langle\sigma\rangle \circlearrowright X$

X = disjoint union of orbits under $\langle\sigma\rangle = H$

We find them by writing σ as a product of disjoint cycles:

$$\sigma = (i_1 \dots i_{k_1})(i_{k_1+1} \cdots i_{k_2}) \cdots (i_{k_s+1}, \cdots i_n)$$

Then $_H\backslash X = (\{i_1 \cdots i_{k_1}\}; \{i_{k_1+1}, \cdots i_{k_2}\}, \cdots \{i_{k_s+1}, \cdots, i_n\})$

Eg: $n=5$ $\quad \sigma = (123)(4)(5) \Rightarrow {}_{\langle\sigma\rangle}\backslash X = \{\{1,2,3\}, \{4\}, \{5\}\}$

$\mathrm{Stab}_{\langle\sigma\rangle}(1) = \mathrm{Stab}_{\langle\sigma\rangle}(2) = \mathrm{Stab}_{\langle\sigma\rangle}(3) = \{e\}$, $\mathrm{Stab}_{\langle\sigma\rangle}(4) = \mathrm{Stab}_{\langle\sigma\rangle}(5) = \langle\sigma\rangle$

$$\underset{5}{\overset{\shortparallel}{|X|}} = \sum_{\alpha \in {}_H\backslash X} |H| / |\mathrm{Stab}_H(x_\alpha)| \overset{x_\alpha = 1,4,5}{=} \frac{3}{1} + \frac{3}{3} + \frac{3}{3} = 3+1+1=5 \checkmark$$

Next: $|{}_H\backslash X| = \#$ cycles in σ (including 1-cycles)

$$\underset{i=1,\dots,o(\sigma)-1}{|X^{\sigma^i}|} = \# \text{ 1-cycles in } \sigma^i = \sum_{j|i} \#(\text{cycles of length } j) \cdot j$$

Burnside: $\#$ cycles in $\sigma = |{}_{\langle\sigma\rangle}\backslash X| = \frac{1}{|\sigma|} \sum_{g \in \langle\sigma\rangle} |X^g|$

$$= \frac{1}{\mathrm{lcm}(\text{cycle length in } \sigma)} \left(n + \sum_{i=1}^{o(\sigma)-1} \sum_{j|i} j \cdot \# \text{ cycles of length } j \right)$$

Eg $n=5$ $\quad \sigma = (123)(4)(5)$ $\quad o(\sigma) = 3 = \mathrm{lcm}(3,1,1)$

$$3 \overset{?}{=} \frac{1}{3}\Big(5 + (\underset{\substack{i=1\\j=1}}{1 \cdot 2}) + (\underset{\substack{i=2\\j=1}}{1 \cdot 2} + \underset{\substack{i=2\\j=2}}{2 \cdot 0})\Big) = \frac{9}{3} = 3 \checkmark$$

§2 Applications

① Pick n & an composition of $n = (a_1, \ldots, a_r)$ $a_1 + \cdots + a_r = n$ $a_i \in \mathbb{Z}_{>0}$

$\Rightarrow X$ - set of all partitions $P_1 \sqcup \cdots \sqcup P_r$ of $\{1, \ldots, n\}$ with $|P_i| = a_i$.

Eg $n = 7 = 3+2+2$ $|P_1| = 3$, $|P_2| = 2$, $|P_3| = 2$

Then $S_n \circlearrowright X$ & this action is transitive (one orbit!)

$\mathrm{Stab}_{S_n}(x) \cong S_{a_1} \times S_{a_2} \times \cdots \times S_{a_r}$ (with coordwise multiplication)

Eg: $\mathrm{Stab}_{S_n}(\{1,2,3\} \sqcup \{4,5\} \sqcup \{6,7\}) \simeq S_3 \times S_2 \times S_2$

$$\Rightarrow |X| = \frac{|S_7|}{|\mathrm{Stab}_{S_n}(x)|} = \frac{7!}{3!\,2!\,2!}$$

In general, we get the formula for the multinomial coeff!

$$\boxed{\begin{array}{l}\#\text{ partitions of } \{1, \ldots, n\} \text{ with } r \text{ parts of sizes } (\lambda_1, \ldots, \lambda_r) = \dfrac{n!}{\lambda_1! \cdots \lambda_r!} \\ (|P_i| = \lambda_i \ \forall i \quad \lambda_1 + \cdots + \lambda_r = n)\end{array}}$$

② Let $p > 0$ be a prime number

Def: A group G is said to be a p-group if $G = p^k$ for some $k \in \mathbb{Z}_{\geq 1}$.

Eg: $G = \mathbb{Z}/p^k\mathbb{Z}$ is a p-group.

Lemma: Let G be a p-group acting on a finite set X. Then

$$|X| \equiv |X^G| \mod p.$$

Here: $X^G = \bigcap_{g \in G} X^g = \{x \in X : g \cdot x = x \ \forall g \in G\}$

Proof: By (*) on page 1 $|X| = \sum_{\alpha \in G\backslash X} |G \cdot x_\alpha| = |X^G| + \underbrace{\sum_{\substack{\alpha \in G\backslash X \\ |G \cdot x_\alpha| > 1}} |G \cdot x_\alpha|}_{p|}$

orbits of size 1

$\Rightarrow |X| \equiv |X^G| \mod (p)$.

since $1 < |G \cdot x_\alpha| \mid |G| = p^k$

③ Prop: Given any prime $p \in \mathbb{Z}_{\geq 2}$ & $m \in \mathbb{Z}_{\geq 1}$, we have $\binom{p^r m}{p^r} \equiv m \pmod p$

Proof: (1) Induct on r

(2) Use group actions! Take $G = \mathbb{Z}/p^r\mathbb{Z}$, $X = \{x_1, \dots, x_m\}$ any set with m elements.

- E = set of all p^r element subsets of $G \times X$. $\rightsquigarrow |E| = \binom{p^r m}{p^r}$
- $G \circlearrowright G \times X$ by $\sigma(g, x) = (\sigma \cdot g, x)$

So $G \circlearrowright E$ by $\sigma\{e_1, \dots, e_{p^r}\} = \{\sigma(e_1), \sigma(e_2), \dots, \sigma(e_{p^r})\} \in E$ ✓

(axioms for left action are satisfied)

- By Lemma: $\boxed{|E| \equiv \#(\text{of orbits with exactly one element}) \pmod p}$

Let's count how many such orbits we have: 2^{nd} entry of each member of the orbit is fixed!

$\Rightarrow$ Orbits are $\{(g, x_1) : g \in G\}$, $\{(g, x_2) : g \in G\}, \dots \{(g, x_m) : g \in G\}$

$\Rightarrow$ m of them!

We get $\binom{p^r m}{m} = |E| \equiv m \pmod p$ □

Obs: This will be used in one of the Sylow Theorems.

§3 Actions of G on itself

① Left Multiplication: $L: G \longrightarrow \mathrm{Aut}_{set}(G)$
$g \longmapsto L_g$

where $L_g(x) = gx \quad \forall x$

② Right Multiplication $R: G \longrightarrow \mathrm{Aut}_{set}(G)$
$g \longmapsto R_g$

Where $R_g(x) = x \cdot g^{-1} \quad \forall x$

(Why inverse is necessary? We want $R_{g_1} \circ R_{g_2} = R_{g_1 g_2}$

$(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$ whereas $g_1 g_2 \neq g_2 g_1$ unless G is abelian.

Similar situation:

$$G \circlearrowright X \rightsquigarrow G \circlearrowright \mathrm{Fun}(X, Y) \text{ via } (g\cdot f)(x) = f(g^{-1}\cdot x)$$
Y set $\qquad f \in \mathrm{Fun}(X,Y)$

$$(g_1\cdot(g_2\cdot f))(x) = (g_2\cdot f)(g_1^{-1}x) = f(g_2^{-1}g_1^{-1}x) = f((g_1g_2)^{-1}x) = (g_1g_2\cdot f)(x)$$

$(e\cdot f)(x) = f(x)$ ✓

③ Conjugation (HW1) $\qquad C: G \longrightarrow \mathrm{Aut}_{set}(G)$, $g \longmapsto C_g$

where $C_g(x) = gxg^{-1}$ $\forall x$.

§4. More applications:

Consider $G \circlearrowright G$ by conjugation.

For $x \in G$, the stabilizer $= \{g \in G \mid gxg^{-1} = x\}$ is also called the centralizer of x. We denote it by $Z_G(x)$.

- G-orbits under conjugation are called conjugacy classes. Write the set of all these classes by $\mathcal{C}$.

Obs: For each $g\in G$, the set of elements of G fixed under C_g is $Z_G(g)$

(Indeed: $hgh^{-1} = g \iff hg = gh \iff g^{-1}hg = h \iff ghg^{-1} = h$)

By our counting lemmas:

- $$|G| = \sum_{\alpha\in\mathcal{C}} |G\cdot x_\alpha| = \sum_{\alpha\in\mathcal{C}} \left|\frac{G}{Z_G(x_\alpha)}\right| = \sum_{\alpha\in\mathcal{C}} [G : Z_G(x_\alpha)].$$
- $$\underbrace{|\mathcal{C}|}_{\#\text{ conj. classes}} = \frac{1}{|G|}\sum_{\alpha\in\mathcal{C}} |G^{x_\alpha}| \underset{\text{by Obs}}{=} \underbrace{\frac{1}{|G|}\sum_{\alpha\in\mathcal{C}} |Z_G(x_\alpha)|}_{\text{average \# of elements in a centralizer}}$$