Lecture 7 : Sylow Theorems

Last week:
- Defined $\underset{gp}{G} \circlearrowright \underset{set}{X}$ via $G \longrightarrow Aut_{set}(X)$ gp hom.
  $g \longmapsto (x \longmapsto g\cdot x)$
- $G\cdot x \subseteq X$ orbit $\forall x \in X$, $Stab_G(x) < G$ stabilizer $\forall x \in X$
- $X^g \subseteq X$ fixed pt set $\forall g \in G$
- Examples: ① $S_n \circlearrowright \{1,..,n\}$, ③ $G \circlearrowright G$ left mult $x \longmapsto g\cdot x$
  right — $x \longmapsto x\cdot g^{-1}$
  ② $D_n \circlearrowright \mathbb{R}^n \setminus \{(0,0)\}$ conjugation $x \longmapsto gxg^{-1}$
- Discussed counting lemmas:

(1) $|G| = |Stab_G(x)|\,|G\cdot x| \quad \forall x \in G$

(2) $|X| = \sum_{\alpha \in {}_G\backslash X} |G\cdot x_\alpha| = \sum_{\alpha \in {}_G\backslash X} \frac{|G|}{|Stab_G(x_\alpha)|}$

(3) Burnside $\left|{}_G\backslash X\right| = \frac{1}{|G|} \sum_{g \in G} |X^g|$. ($G$ finite)

- Applications ① $\binom{p^r m}{p^r} \equiv m \pmod p$

② $|G| = p^k$ (p-group) $\Rightarrow |X| \equiv |X^G| \pmod p$ where $X^G = \bigcap_{g\in G} X^g$

§1. Sylow Theorems:

Fix $p>0$ prime & write $n = p^r m$ with $(m:p)=1$.
Let G be a group of order n.

Definition: A subgroup $P<G$ of order p^r is called a Sylow p-subgp of G

Sylow Theorems: (A) Sylow p-subgroups exist.

(B1) If $H<G$ is a p-group, then there exists a Sylow p-subgroup $P<G$ with $H \subseteq P$.

(B2) Any two Sylow p-subgroups $P, Q < G$ are conjugate to each other (ie $\exists\, g \in G$ with $Q = gPg^{-1}$)

(C) Let n_p = number of Sylow p-subgroups of G. Then (i) $n_p \equiv 1 \pmod p$
(ii) $n_p \mid m$

We will prove these theorems using group actions. An alternative proof will be discussed in HW3

§2 Proof of Sylow Thm (A):

Let $\mathcal{E} = \{Y \subset G \text{ subset} : |Y| = p^r\}$

$G \circlearrowright \mathcal{E}$ induced by left multiplication action $G \circlearrowright G$, ie:

Given $g \in G$ & $Y = \{y_1, \ldots, y_{p^r}\} \in \mathcal{E} \rightsquigarrow \boxed{g \cdot Y} = \{g \cdot y_1, \ldots, g \cdot y_{p^r}\} \in \mathcal{E}$

<u>Claim</u>: There exists $X \in \mathcal{E}$ whose orbit has cardinality <u>not</u> divisible by p.

Pf/ By the Counting Lemma 1: $|\mathcal{E}| = \sum_{\alpha \in G\backslash\mathcal{E}} |G \cdot x_\alpha|$

By Application 1 $|\mathcal{E}| = \binom{p^r m}{p^r} \equiv m \pmod p$

But $m \not\equiv 0 \mod (p)$ so $p \nmid |G \cdot x_\alpha|$ for some α. □

- Let $H_X = \mathrm{Stab}_G(X) < G$ Then $|G \cdot X| = \dfrac{|G|}{|H_X|} \not\equiv 0 \mod (p)$

So $p^r \mid |H_X|$.

- To finish, choose $x_0 \in X$ and define $\varphi: H_X \longrightarrow X$ (set map)
$g \longmapsto g \cdot x_0$ (by definition of stabilizer)

The map φ is <u>injective</u> since $g x_0 = h x_0$ in $G \Rightarrow g = h$.

$\Rightarrow |H_X| \leqslant |X| = p^r$

Thus $|H_X| = p^r$ and hence H_X is a Sylow p-subgroup of G □

§3 Proof of Sylow Theorem (B):

(B1) Let $H < G$ be a p-subgroup of G, so $|H| = p^k$ with $k \leq r$.

Let Q be a Sylow p-subgroup of G (which exists by Thm (A))

We consider the action of H on $X := G/Q$: $h \cdot gQ = (hg)Q$.

By Application ② $|X^H| \equiv |X| \pmod p$. As $|X| = m \not\equiv 0 \pmod p$ then $X^H \neq \emptyset$, so $\exists\, gQ \in X^H$, that is $hgQ = gQ \quad \forall h \in H$

We conclude $g^{-1}hg \in Q \quad \forall h \in H$ so $\boxed{H \subseteq gQg^{-1}}$

• Take $\boxed{P = gQg^{-1}}$. Since $|gQg^{-1}| = |Q| = p^r$, then P is a Sylow p-subgroup of G & $H \subseteq P$.

(B2) To prove B2 we take $H = P$ and Q any p-Sylow subgroup.
By (B1) we can find $g \in G$ with $P \subseteq gQg^{-1}$
But since $|P| = p^r = |gQg^{-1}|$, then $P = gQg^{-1}$ so Q & P are conjugate to each other. □

<u>Obs:</u> The proof of Sylow Thm (A) is not constructive (in practice), but the proof of Sylow Thm (B1) yields a potential algorithm. Start with an element x of order p^k for some $k > 0$ & set $H = \langle x \rangle$. Then, try to add elements to it (again, of order a power of p) to extend H to a Sylow p-subgroup of G.

§4. Proof of Sylow Thm (C):

Let $\mathcal{S}$ be the set of all Sylow p-subgroups of G.
By Thm (A), $\mathcal{S} \neq \emptyset$. By definition, $|\mathcal{S}| = n_p$.

By Thm (B), we know that $G \circlearrowright \mathcal{S}$ by conjugation is transitive
$(x \cdot Q = xQx^{-1})$.

Consider $P \in \mathcal{S}$ & restrict the action to $P \circlearrowright \mathcal{S}$.

<u>Claim:</u> $\mathcal{S}^P = \{P\}$

By Application ② $|\mathcal{S}^P| \equiv |\mathcal{S}| \pmod p$. Thus, Claim yields

$$\boxed{n_p = |\mathcal{S}| \equiv |\mathcal{S}^P| = 1 \mod p.}$$

<u>Proof of Claim</u>: Fix $Q \in \mathcal{S}^P$, ie Q is a Sylow p-subgroup of G with $xQx^{-1} = Q \quad \forall x \in P$.

Define $\boxed{N_Q} = \{g \in G : gQg^{-1} = Q\} < G$ (normalizer of Q in G)

$= \mathrm{Stab}_G(Q)$ under action by conjugation

Then $Q, P \subset N_Q$ & $|N_Q| \mid |G| = p^r m$ so P & Q are two Sylow p-subgroups of N_Q.

By (B2) P & Q are conjugate in N_Q so $\exists\, x \in N_Q$ with $xQx^{-1} = P$. But $xQx^{-1} = Q$ since $x \in N_Q$.

We conclude $\boxed{P = Q}$ $\square$

- It remains to show that $n_p \mid m$. To do so, we consider $G \curvearrowright \mathcal{S}$ by conjugation (This action is transitive by (B2)). Choose $P \in \mathcal{S}$ & Use Counting Lemma 1:

$$n_p = |\mathcal{S}| = |G \cdot P| = \frac{|G|}{|\mathrm{Stab}_G(P)|} = \frac{|G|}{|N_P|}$$

Since $P < N_P < G \Rightarrow |N_P| = p^r m'$ for some $m' \mid m$.

Then $n_p = \frac{m}{m'}$ & so $\boxed{n_p \mid m}$. $\square$

<u>§5 An example</u>:

Fix $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$ be the field with 5 elements.

$G = GL_2(\mathbb{F}_5) = \{$ invertible 2×2 matrices with entries from $\mathbb{F}_5\}$

<u>Claim 1</u>: $|G| = 480$

Pf/ # choices for 1^{st} column $= |\mathbb{F}_5^2 \setminus \{[\begin{smallmatrix}0\\0\end{smallmatrix}]\}| = 25-1 = 24$

choices for 2^{nd} column $= |\mathbb{F}_5^2 \setminus \{\underset{\in \mathbb{F}_5}{\lambda} \cdot 1^{st} \text{ column}\}| = 25-5 = 20$

$480 = 24 \cdot 20$ ✓ $\square$

Since $480 = 2^5 \cdot 3 \cdot 5$, then we have Sylow p-subgroups for $p = 2, 3, 5$.

- A Sylow 5-subgroup : $P = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad x \in \mathbb{F}_5 \right\} = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$

$\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} \in P, \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \in P \right)$

$n_5 \equiv 1 \pmod 5$ & $n_5 \mid 2^5 3$

$n_5 \equiv 2^k 3^j \equiv 2^k (-2)^j \equiv (-1)^j 2^{k+j} \equiv 1 \mod (5)$

$j = 0, 1, \quad k = 0, \dots, 5$

If $j = 0$, then $k = 0, 4$ so $n_5 = 1$ or 16

If $j = 1$, then $k = 1, 5$ so $n_5 = 6$ or 96.

Another Sylow subgroup $P' = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \quad x \in \mathbb{F}_5 \right\} = \left\langle \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle$ so $n_5 \neq 1$.

Use Thm (B2) & conjugate the generator of P

$$\frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d-c & -b+a \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-ac & -bc & -ab+a^2 & +ba \\ cd-c^2 & -cd & -bc+ad & +ac \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc-ac & a^2 \\ -c^2 & ad-bc+ac \end{bmatrix} = B.$$

All Sylow 5-subgroups of G are $\langle B \rangle$

If $ad-bc = 1$, then $B = 1 \begin{bmatrix} 1-ac & a^2 \\ -c^2 & 1+ac \end{bmatrix}$ so 24 options

If $ad-bc = 2$ —— $B = \begin{bmatrix} 1-3ac & 3a^2 \\ -3c^2 & 1+3ac \end{bmatrix}$ so 24 options

We don't need to consider $\det A = -1, \overset{-2}{3}$ because -1 has a square root modulo 5. $\lambda A \begin{bmatrix}1&1\\0&1\end{bmatrix}(\lambda A)^{-1} = A\begin{bmatrix}1&1\\0&1\end{bmatrix}A^{-1}$ & $\det \lambda A = \lambda^2 \det A$.

So at most 48 generators B, we get $n_5 \leq 48$

Only options left $= 6$ or 16. Start with $\det A = 1$.

- $c = 0$ & any $a \neq 0$ gives P, $\left(B = \begin{bmatrix} 1-ac & a^2 \\ -c^2 & 1+ac \end{bmatrix} \right)$ } TOTAL $= 2$
- $a = 0$ & any $c \neq 0$ gives P'

Next we compute the remaining combinations & $\{B^2, B^3, B^4\}$.

$a \backslash c$	1	2	3	4	
1	$\begin{bmatrix}0&1\\4&2\end{bmatrix}$	$\begin{bmatrix}4&1\\1&3\end{bmatrix}$	$\begin{bmatrix}3&1\\1&4\end{bmatrix}$	$\begin{bmatrix}2&1\\4&0\end{bmatrix}$	$\rightsquigarrow 3$
2	$\begin{bmatrix}4&4\\4&3\end{bmatrix}$	$\begin{bmatrix}2&4\\1&0\end{bmatrix}$	$\begin{bmatrix}0&4\\1&2\end{bmatrix}$	$\begin{bmatrix}3&4\\4&4\end{bmatrix}$	$\rightsquigarrow 1$
3	$\begin{bmatrix}3&4\\4&4\end{bmatrix}$	$\begin{bmatrix}0&4\\1&2\end{bmatrix}$	$\begin{bmatrix}2&4\\1&0\end{bmatrix}$	$\begin{bmatrix}4&4\\4&3\end{bmatrix}$	
4	$\begin{bmatrix}2&1\\4&0\end{bmatrix}$	$\begin{bmatrix}3&1\\1&4\end{bmatrix}$	$\begin{bmatrix}4&1\\1&3\end{bmatrix}$	$\begin{bmatrix}0&1\\4&2\end{bmatrix}$	TOTAL $= 4$ $\langle B \rangle$

$\begin{bmatrix}0&1\\4&2\end{bmatrix}^2 = \begin{bmatrix}4&2\\3&3\end{bmatrix}$, $\begin{bmatrix}0&1\\4&2\end{bmatrix}^3 = \begin{bmatrix}3&3\\2&4\end{bmatrix}$, $\begin{bmatrix}0&1\\4&2\end{bmatrix}^4 = \begin{bmatrix}2&4\\1&0\end{bmatrix}$, $\begin{bmatrix}0&1\\4&2\end{bmatrix}^5 = I_5$

$\begin{bmatrix}4&1\\1&3\end{bmatrix}^2 = \begin{bmatrix}2&2\\2&0\end{bmatrix}$, $\begin{bmatrix}4&1\\1&3\end{bmatrix}^3 = \begin{bmatrix}0&3\\3&2\end{bmatrix}$, $\begin{bmatrix}4&1\\1&3\end{bmatrix}^4 = \begin{bmatrix}3&4\\4&4\end{bmatrix}$

$\begin{bmatrix}3&1\\1&4\end{bmatrix}^2 = \begin{bmatrix}0&2\\2&2\end{bmatrix}$, $\begin{bmatrix}3&1\\1&4\end{bmatrix}^3 = \begin{bmatrix}2&3\\3&0\end{bmatrix}$, $\begin{bmatrix}3&1\\1&4\end{bmatrix}^4 = \begin{bmatrix}4&4\\4&3\end{bmatrix}$

$\begin{bmatrix}2&1\\4&0\end{bmatrix}^2 = \begin{bmatrix}0&2\\2&1\end{bmatrix}$, $\begin{bmatrix}2&1\\4&0\end{bmatrix}^3 = \begin{bmatrix}2&0\\0&2\end{bmatrix}$, $\begin{bmatrix}2&1\\4&0\end{bmatrix}^4 = \begin{bmatrix}4&2\\2&0\end{bmatrix}$

$\begin{bmatrix}0&4\\1&2\end{bmatrix}^2 = \begin{bmatrix}4&3\\2&3\end{bmatrix}$, $\begin{bmatrix}0&4\\1&2\end{bmatrix}^3 = \begin{bmatrix}3&2\\3&4\end{bmatrix}$, $\begin{bmatrix}0&4\\1&2\end{bmatrix}^4 = \begin{bmatrix}2&1\\3&0\end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}^3 = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix} \rightsquigarrow (a,c) = (2,3)$$

$$\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix} \rightsquigarrow (a,c) = (1,1)$$

We repeat the computation with $\det B = 2$.

- $c = 0$ & any $a \neq 0$ gives P $\quad \left(B = \begin{bmatrix} 1-3ac & 3a^2 \\ -3c^2 & 1+3ac \end{bmatrix} \right)$
- $a = 0$ & any $c \neq 0$ — P'

Next we compute the remaining combinations & $\{B^2, B^3, B^4\}$.

$a \backslash c$	1	2	3	4
1	$\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}$	$\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}$	$\begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}$
2	$\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$	$\begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$
3	$\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$
4	$\begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}$	$\begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}$

$$\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix} \rightsquigarrow (a,c) = (1,1) \text{ in TABLE I.}$$

$$\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}^2 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \rightsquigarrow (a,c) = 1,2) \text{ in TABLE I.}$$

$$\begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \rightsquigarrow (a,c) = (1,3) \text{ in TABLE I.}$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix} \rightsquigarrow (a,c) = (1,4) \text{ in TABLE I}$$

$$\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}^2 = \begin{bmatrix} 4 & 4 \\ 4 & 3 \end{bmatrix} \rightsquigarrow (a,c) = (2,1) \text{ in TABLE I}$$

$$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}^2 = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \rightsquigarrow (a,c) = (2,2) \text{ in TABLE I}$$

$$\begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix} \rightsquigarrow (a,c) = (2,3) \text{ in TABLE I}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix} \rightsquigarrow (a,c) = (2,4) \text{ in TABLE I}$$

Conclusion: $\boxed{n_5 = 6.}$

• A Sylow 3-subgroup $P = \left\langle \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\rangle$ (need an order 3 matrix)

A is companion matrix of $p(t) = t^2 + t + 1$ & $p(t)(t-1) = t^3 - 1$.

$n_3 \equiv 1 \pmod 3$ & $n_3 \mid 2^5 5$

$n_3 = 2^k 5^j \equiv (-1)^{k+j} \pmod 3$ so $k+j$ is even

$j = 0, 1$, $k = 0, \dots, 5$

So $n_3 = \underbrace{\underset{k=0}{1}, \underset{2}{4}, \underset{4}{16}}_{j=0}, \underbrace{\underset{k=1}{6}, \underset{3}{40}, \underset{5}{160}}_{j=1} \rightsquigarrow$ $\boxed{n_3 = 1, 4, 6, 16, 40 \text{ or } 160}$

Another Sylow 3-subgroup of G is $P' = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right\rangle$ so $n_3 \neq 1$

Use Thm (32) & conjugate P' to get the full list (Exercise)

• A Sylow 2-subgroup $P = \left\{ \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} 0 & \mu_1 \\ \mu_2 & 0 \end{bmatrix} \ \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{F}_5^* \right\}$

Check it's a group!

$\cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 0 & \mu_1 \\ \mu_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 \mu_1 \\ \lambda_2 \mu_2 & 0 \end{bmatrix} \in P$ ✓ $\quad \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix}$ ✓

$\cdot \begin{bmatrix} 0 & \mu_1 \\ \mu_2 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \mu_1 & 0 \\ 0 & \lambda_2 \mu_2 \end{bmatrix} \in P$ ✓ $\quad \begin{bmatrix} 0 & \mu_1 \\ \mu_2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{\mu_2} \\ \frac{1}{\mu_1} & 0 \end{bmatrix}$ ✓

$n_2 \equiv 1 \pmod 2$ & $n_2 \mid 15 \rightsquigarrow n_2 = 1, 3, 5, 15$

$$\frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_1 d & -\lambda_1 b \\ -\lambda_2 c & \lambda_2 a \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} \lambda_1 ad - \lambda_2 bc & ab(\lambda_2 - \lambda_1) \\ cd(\lambda_1 - \lambda_2) & \lambda_2 ad - \lambda_1 bc \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} \lambda_1(ad-bc) + bc(\lambda_1 - \lambda_2) & -ab(\lambda_1 - \lambda_2) \\ cd(\lambda_1 - \lambda_2) & \lambda_2(ad-bc) - bc(\lambda_1 - \lambda_2) \end{bmatrix}$$

$$\frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & \mu_1 \\ \mu_2 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -\mu_1 c & \mu_1 a \\ \mu_2 d & -\mu_2 b \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} -\mu_1 ac + \mu_2 bd & a^2 \mu_1 - b^2 \mu_2 \\ -\mu_1 c^2 + \mu_2 d^2 & \mu_1 ac - \mu_2 bd \end{bmatrix}$$

This calculation will help compute n_2 (at least it will show $n_2 \neq 1$)