Lecture 9: Semidirect products

Last Time: Used Sylow Theorems to study non-simple groups of a fixed order

- Classified groups of order p^2 with $p>0$ prime

TODAY: New tool for classifying groups.

§1 Motivation:

- **HW1:** Saw all groups of order ≤ 5 are abelian; S_3, D_3 are not.
- **Goal:** Classify groups G of order $6 = 2\cdot 3$

By Sylow Thm: $\left.\begin{array}{l} n_2 \equiv 1 \mod (2) \\ n_2 \mid 3 \end{array}\right\} n_2 = 1 \text{ or } 3$; $\left.\begin{array}{l} n_3 \equiv 1 \mod 3 \\ n_3 \mid 2 \end{array}\right\} \Rightarrow \boxed{n_3 = 1}$

$Syl_3(G) = \{Q\}$ with $Q = \langle h \rangle \simeq \mathbb{Z}/3\mathbb{Z}$; $Syl_2(G) \ni P = \langle g \rangle \simeq \mathbb{Z}/2\mathbb{Z}$

CASE 1: $n_2 = 1$ Then, $Syl_2(G) = \{P\}$

Since $P, Q \lhd G$ & $P \cap Q = \{e\} \underset{HW1}{\Longrightarrow} P$ & Q mutually commute.

$$\begin{array}{ccccc} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \simeq & P \times Q & \longrightarrow & G \\ (k, \ell) \mapsto & (g^k, h^\ell) & \longrightarrow & g^k h^\ell \end{array}$$

is an injective gp homomorphism

$|G| = |P \times Q| = 6$, so iso

$$G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

CASE 2: $n_2 = 3$ Then, $Syl_2(G) = \{\overset{P}{\overset{\shortparallel}{P_1}}, P_2, P_3\}$

Obs: h & g cannot commute (Otherwise $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ so $n_2 = 1$)

$G = \{1, h, h^2, g, gh, gh^2\}$ is a group so $hg = gh^2$ so $ghg^{-1} = ghg = h^2$

Conclude: $G \simeq D_3$ where $h \longmapsto \rho$, $g \longmapsto s$ (same relations!) & $D_3 \simeq S_3$, $\rho \mapsto (123)$, $s \mapsto (12)$

$[(12)(123)(12) = (132)]$

- We can view the construction more generally!

① We let P act on $Q \lhd G$ by conjugation:

$$\begin{array}{rcll} \alpha : P & \longrightarrow & Aut_{Set}(Q) & \text{gp hom} \\ g^i & \longmapsto & \left(g^i h^j g^{-i} = \begin{cases} h^{2j} & \text{if } i=1 \\ h^j & \text{if } i=0 \end{cases} \right) \end{array}$$

$i=0$: $h^j \xrightarrow{\alpha(e)} h^j$

$i=1$: $h^{j+k} \xrightarrow{\alpha(g)} \underset{\alpha(g)(h^j)\,\alpha(g)(h^k)}{\overset{\shortparallel}{h^{2(j+k)}}}$

So $\alpha : P \longrightarrow Aut_{Gp}(Q)$ is group hom.

② $G = PQ = \{g^i h^j\} = QP$ & $Q \lhd G$ (saw in 3rd Iso Thm)
with $P \cap Q = \{e\}$.

- The map α provides the "commutation relation" between $g \in P$ & $h \in Q$.

§2. Semidirect Products I

We will give 3 equivalent constructions, starting from ②.

Definition: We say a group G is a <u>semi-direct product</u> of two subgroups H&N if

(i) $H \leq G$ & $N \trianglelefteq G$

(ii) $G = HN = \{h.n \;\; h \in H, n \in N\} = NH$ [Write: $G = N \rtimes H$]
\rtimes

(iii) $H \cap N = \{e\}$

Obs: Recall 3rd Iso Theorem:

$H < G$
$N \lhd G$ $\rightsquigarrow$

$$\begin{tikzcd}
 & G \arrow[d, dash, "\vee"] & \\
 & HN = NH \arrow[dl, dash, "\leq"'] \arrow[dr, dash, "\rhd"] & \\
H \arrow[dr, dash, "\rhd"'] & & N \arrow[dl, dash, "\leq"] \\
 & H \cap N &
\end{tikzcd}$$

(= Today)

& $\underset{\text{(today)}}{\dfrac{H}{e}} = \dfrac{H}{H\cap N} \xrightarrow{\sim} \dfrac{HN}{N} = \underset{\text{(today)}}{\dfrac{G}{N}}$

Consequence: $H \simeq G/N$, so for each coset $gN \in G/N$ we can find a representative $\sigma_g \in H$ so that $\sigma_{g_1 g_2} = \sigma_{g_1} \cdot \sigma_{g_2}$.

Example 1: $H \lhd G$, then $G \simeq \underset{\text{(coord.wise structure)}}{N \times H}$ (direct product)

Example 2 $G = \underset{\langle s, \rho \rangle}{\overset{\shortparallel}{D_n}}$ $\quad H = \{e, s\} < G$; $N = \langle \rho \rangle \lhd G$
$\qquad\qquad\qquad\quad\;\; \mathbb{Z}/2\mathbb{Z} \qquad\qquad \mathbb{Z}/n\mathbb{Z}$ $\quad (s\rho^j s^{-1} = \rho^{-j})$

$\Rightarrow D_n \simeq \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$

Example 3: $G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in GL_2(\mathbb{C}) \right\}$ $\quad \mathbb{C} \simeq N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\} \lhd G$

$(\mathbb{C}^*)^2 \simeq H = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in G \right\} < G$

- $H \cap N = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
- $G = HN = NH$ because $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \underset{H}{\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}} \underset{N}{\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}} = \underset{N}{\begin{bmatrix} 1 & b/c \\ 0 & 1 \end{bmatrix}} \underset{H}{\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}}$.

$\Rightarrow G \simeq \mathbb{C} \rtimes (\mathbb{C}^*)^2$

§3. Semidirect Products II:

Fix two groups H & N and a group homomorphism

$$\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) = \{ \text{isos} : N \xrightarrow{\sim} N \}$$

Obs: $\alpha(h_1 h_2) = \alpha_{(h_1)} \circ \alpha_{(h_2)}$, $\alpha(e) = id_N$, $\alpha(h^{-1}) = \alpha(h)^{-1}$

We can use this to define a binary operation of the cartesian product $N \times H$.

$$G = \{ (n, h) : n \in N , h \in H \}$$

$$\boxed{(n_1, h_1) * (n_2, h_2) = (n_1 \cdot \alpha_{(h_1)}(n_2), h_1 h_2) \quad \forall\, n_1, n_2 \in N,\ h_1, h_2 \in H.}$$

Lemma: This operation defines a group structure on G. Write $\boxed{G = N \rtimes_\alpha H}$
(H acts on N via α)

Pf/ We need to check associativity, neutral element & inverses:

① Associativity:

$g_1 = (n_1, h_1)$, $g_2 = (n_2, h_2)$, $g_3 = (n_3, h_3) \in G$

$g_2 \cdot g_3 = (n_2 \cdot \alpha(h_2)(n_3), h_2 h_3)$; $g_1 g_2 = (n_1 \alpha_{(h_1)}(n_2), h_1 h_2)$

$g_1 (g_2 g_3) = (n_1\ \alpha(h_1)(n_2 \alpha_{(h_2)}(n_3)) , h_1(h_2 h_3))$

$= (n_1\ \alpha(h_1)(n_2) \cdot \alpha(h_1 h_2)(n_3) , (h_1 h_2) h_3)$

α gp hom.
$\alpha(h_1)$ gp hom.

$= ((n_1, h_1)(n_2, h_2)) \cdot (h_3, n_3) = (g_1 g_2) g_3$

② Neutral Element: (e_N, e_H)

Why? $(n, h_1) \cdot (n_2, h_2) = (n, h_1)$

$(n_1 \alpha_{(h_1)}(n_2), h_1 h_2) = (n_1, h_1)$ $\forall h_1$ forces $h_2 = e_H$
$\alpha(h_1)(n_2) = e_N$ $\forall h_1$

In particular for $h_1 = e_H$, so $n_2 = e_N$.

Check: $(n_1, h_1)(e_N, e_H) = (n_1\ \underbrace{\alpha_{(h_1)}(e_N)}_{=e_N}, h_1 e_H) = (n_1, h_1)$ ✓

$(e_N, e_H)(n_1, h_1) = (e_N\ \underbrace{\alpha(e_H)}_{id}(n_1), e_H h_1) = (n_1, h_1)$ ✓

③ **Inverses:** $(n,h)^{-1} = (\alpha(h^{-1})(n^{-1}), h^{-1})$

Why? $(n_1,h_1)(n_2,h_2) = (n_1\alpha(h_1)(n_2), h_1h_2) = (e_N, e_H)$ forces $h_2 = h_1^{-1}$

& $\alpha(h_1)(n_2) = n_1^{-1}$ so $n_2 = \alpha(h_1)^{-1}(n_1^{-1}) = \alpha(h_1^{-1})(n_1^{-1})$

Check: $(n,h)(\alpha(h^{-1})(n^{-1}), h^{-1}) = (n\,\underbrace{\alpha(h)(\alpha(h^{-1})}_{id}(n^{-1})), hh^{-1}) = (e_N, e_H)$ ✓

$$(\alpha(h^{-1})(n^{-1}), h^{-1})(n,h) = (\alpha(h^{-1})(n^{-1})\;\alpha(h^{-1})(n), h^{-1}h)$$
$$= (\alpha(h^{-1})(\underbrace{n^{-1}n}_{=e_N}), e_H) = (e_N, e_H)$$ ✓

$\alpha(h^{-1})$ gp hom.

□

Next, we relate $N \rtimes_\alpha H$ to the construction from §2.

Proposition: The maps
$$H \longrightarrow G = N\rtimes_\alpha H, \quad h \longmapsto (e_N, h) \quad ; \quad N \longrightarrow G = N\rtimes_\alpha H, \quad n \longmapsto (n, e_H)$$
are injective group homomorphisms. Furthermore:

(i) $H \leq G$, $\quad N \trianglelefteq G$ (via the injections)

(ii) $NH = G$

(iii) $H \cap N = \{(e_N, e_H) =: e_G\}$

So G is the semidirect product of H & N.

Proof: Check the structures of H & N are compatible with that of G

$\underline{H}$: $(e_N,h_1)(e_N,h_2) = (e_N\,\underbrace{\alpha(h_1)(e_N)}_{=e_N}, h_1h_2) = (e_N, h_1h_2)$

$\Longrightarrow$ $H \xrightarrow{\varphi_H} G$, $h \longmapsto (e_N,h)$ is group homomorphism. Clear: $\mathrm{Ker}(\varphi_H) = \{e_H\}$

$\underline{N}$: $(n_1,e_H)(n_2,e_H) = (n_1\,\underbrace{\alpha(e_H)}_{=id_N}(n_2), e_He_H) = (n_1n_2, e_H)$

$\Longrightarrow$ $N \xrightarrow{\varphi_N} G$, $n \longmapsto (n, e_H)$ is gp hom. Clear: $\mathrm{Ker}(\varphi_N) = \{e_N\}$.

(i) H "$=$" $\mathrm{Im}(\varphi_N) \leq G$

Claim: N "=" $\mathrm{Im}(\varphi_N) \trianglelefteq G$

Pf/ $(n,h)\, N\, (n,h)^{-1} = (n,h)\, N\, (\alpha(h^{-1})(n^{-1}), h^{-1}) = N$

$$(n,h)(n_1,e_H)(\alpha(h^{-1})(n^{-1}),h^{-1}) = (n\,\alpha(h)(n_1),h)(\alpha(h^{-1})(n^{-1}),h^{-1})$$
$$= (n\,\alpha(h)(n_1)\,\underbrace{\alpha(h)(\alpha(h^{-1})(n^{-1}))}_{=n^{-1}},\ hh^{-1}) = (n\,\alpha(h)(n_1)\,n^{-1}, e_H) \in N.$$

(ii) $NH = G$ by construction

$$(n,e_H)(e_N,h) = (n\,\underbrace{\alpha(e_H)(e_N)}_{=e_N}, e_H h) = (n,h)$$

(iii) $N \cap H = \{(n,h) : (n,e_H) = (e_N,h)\} = \{(e_N,e_H) = e_G\}$. $\square$

Conversely the construction $N \rtimes H$ always arises from some $N \rtimes_\alpha H$.

Prop: Given $G = N \rtimes H$, we set
$$\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N)$$
$$h \longmapsto (n \longmapsto hnh^{-1})$$

Then, $\Phi : N \rtimes_\alpha H \longrightarrow G$, $(n,h) \longmapsto nh$ is an isomorphism of groups

Proof: It is easy to check α is gp homomorphism, so $N \rtimes_\alpha H$ is well-defined.

Claim 1: Φ is a gp homomorphism.

$$\Phi((n_1,h_1)(n_2,h_2)) = \Phi((n_1\alpha(h_1)(n_2), h_1h_2))$$
$$= \underbrace{n_1\alpha(h_1)(n_2)}_{\in N}\ \underbrace{h_1h_2}_{\in H}$$
$$\overset{\text{def of } \alpha}{=} n_1(h_1n_2h_1^{-1})h_1h_2 = n_1h_1n_2h_2$$
$$= \Phi(n_1,h_1)\,\Phi(n_2,h_2).$$

Claim 2 $\mathrm{Ker}\,\Phi = \{(e_N,e_H)\}$ (because $H \cap N = \{e\}$)

Claim 3: $\mathrm{Im}\,\Phi = G$ (because $NH = G$)

By the 3 Claims, Φ is gp isomorphism. $\square$

Obs: Even though conjugation gives $N \rtimes H$, we might be able to use a different $\alpha' : H \longrightarrow \mathrm{Aut}_{Gp} N$. since different gp hom can give rise to isomorphic gps.

Summary: (not 1-to-1 !)

$$\begin{array}{ccc} \text{Semidirect Products of } N \,\&\, H & \longleftrightarrow & \text{Gp hom } \alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) \\ N \rtimes_\alpha H & \longleftarrow\!\shortmid & \alpha \end{array}$$

Obs: These constructions have been generalized for Hopf algebras. ("Quantum Double Constructions")

- **Next Time:** 3rd Characterization (involving short exact sequences)