Lecture 11: Composition Series & Zassenhaus Lemma

§1 Composition Series

Recall: A group S is called simple if $\{e\}$ & S are the only normal subgroups of S

Examples: A_n $n \geq 5$ are simple (next week)

$\mathbb{Z}/p\mathbb{Z}$ $p > 0$ prime are simple

$PSL_n = SL_n / Z(SL_n)$ are simple

Def: A composition series of a group G is a finite sequence of subgroups of G

$$\Sigma: \quad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = \{e\}$$

such that $G_{j+1} \lhd G_j$ is normal for all $j = 0, \ldots, k-1$.

The successive quotients: $gr_i(G) := G_i / G_{i+1}$ $\quad 0 \leq i \leq k-1$.

(Other notation: $gr_i^{\Sigma}(G)$ if Σ is not clear from context.)

Def 2: A composition series Σ' is said to refine Σ (or be finer than Σ) if Σ is obtained from Σ' by omitting some terms.

More precisely: $\Sigma': \quad G = G'_0 \supseteq \cdots \supseteq G'_m = \{e\}$

$\Sigma: \quad G = G_0 \supseteq \cdots \supseteq G_n = \{e\}$

Σ' is finer than Σ if $n \leq m$ and there exists an order-preserving injective map $\Phi: \{0, \ldots, n\} \longrightarrow \{0, \ldots, m\}$ with $G_j = G'_{\phi(j)}$ $\forall j$.

Ex. 1 $\Sigma_1: G = \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/3\mathbb{Z} \supseteq \{e\}$ no refinement, only 1 coarsening

$\Sigma_2: G = \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$

$gr_0^{\Sigma_1}(G) = \mathbb{Z}/6\mathbb{Z} / \mathbb{Z}/3\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} = gr_1^{\Sigma_2}(G)$, $gr_1^{\Sigma_1} = \mathbb{Z}/3\mathbb{Z} = gr_0^{\Sigma_2}(G)$.

$\Sigma_0: \mathbb{Z}/6\mathbb{Z} \supseteq \{e\}$, Σ_1 refines Σ via $\phi: \{0,1\} \to \{0,1,2\}$ $\phi(0)=0, \phi(1)=2$.

Remark: In general, a series obtained from a composition series Σ' by omitting some terms is NOT a composition series since for $j > i+1$, G'_j is not in general a normal subgroup of G'_i.

Ex 1: $G = D_4 \supseteq \langle \rho^2, s\rangle \supseteq \langle s\rangle \supseteq \{e\}$ $\quad \left(\begin{matrix} s\rho^{i} = \rho^{-i}s \\ s^{-1} = s,\ \rho^4 = 1\end{matrix}\right)$

$G_1 = \langle \rho^2, s\rangle \lhd G$ $\qquad \rho\rho^2\rho^{-1} = \rho^2 \in G_1 \qquad \rho s \rho^{-1} = \rho^2 s \in G_1$

$\qquad s\rho^2 s^{-1} = \rho^2 \in G_1 \qquad sss^{-1} = s \in G_1$

$G_2 = \langle s\rangle \lhd \langle \rho^2, s\rangle$ $\qquad \rho^2 s \rho^{-2} = \rho^2\rho^2 s = \rho^4 s = s \in G_1$

$\qquad sss^{-1} = s \in G_1$

$G_3 = \{e\} \lhd \langle s\rangle$

$$gr_0(D_4) = \frac{\langle \rho, s\rangle}{\langle \rho^2, s\rangle} \cong \frac{\langle \rho\rangle}{\langle \rho^2\rangle} \simeq \mathbb{Z}/2\mathbb{Z}$$

$$gr_1(D_4) = \frac{\langle \rho^2, s\rangle}{\langle s\rangle} \cong \langle \rho^2\rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

$$gr_2(D_4) = \frac{\langle s\rangle}{\langle e\rangle} \simeq \mathbb{Z}/2\mathbb{Z}$$

- We can't omit $\langle \rho^2 s\rangle$ and have a composition series because $\langle s\rangle \ntriangleleft D_4$.
- We can omit $\langle s\rangle$ and get a comp series

$$\Sigma_2: \quad D_4 \supseteq \langle \rho^2, s\rangle \supseteq \{e\}$$

$$gr_0^{\Sigma_2}(D_4) \cong \mathbb{Z}/2\mathbb{Z}, \quad gr_1^{\Sigma_2}(D_4) = \langle \rho^2, s\rangle = D_2.$$

§2 Schreier's Theorem & Zassenhaus' Lemma

We have a notion of equivalence of composition series

Fix $\quad \Sigma_1: \quad G = G_0 \supseteq \dots \supseteq G_m = \{e\}$

$\qquad \Sigma_2: \quad H = H_0 \supseteq \dots \supseteq H_n = \{e\}$ $\quad$ Two composition series

Def. We say Σ_1 & Σ_2 are equivalent if

(i) $m = n$

(ii) $\exists\ \sigma \in S_n = \mathrm{Aut}_{Set}(\{0, \dots, n-1\})$ such that $gr_i^{\Sigma_1}(G) = gr_{\sigma(i)}^{\Sigma_2}(H)$. $\forall i$

Ex: $G = \mathbb{Z}/4\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$
$H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$
are equivalent ($\sigma = id$)

Theorem (Schrier) Let Σ_1 & Σ_2 be two composition series of a group G. Then, there exist composition series Σ_1' & Σ_2' finer than Σ_1 & Σ_2, respectively such that Σ_1' & Σ_2' are equivalent!

[Interpretation : any two composition series have a "common refinement", up to equivalence]

Ex: $G = \mathbb{Z}/6\mathbb{Z}$

$\Sigma_1 = \Sigma_1'$: $\mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/3\mathbb{Z} \supseteq \{e\}$

$\Sigma_2 = \Sigma_2'$: $\mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$

graded pieces : $\mathbb{Z}/2\mathbb{Z}$ & $\mathbb{Z}/3\mathbb{Z}$

Σ_1 & Σ_2 are equivalent via $\sigma \in S_2 : 0 \mapsto 1,\ 1 \mapsto 0$ $\quad gr_0^{\Sigma_1} = gr_1^{\Sigma_2}$, $gr_1^{\Sigma_1} = gr_0^{\Sigma_2}$.

Proof: Write $\Sigma_1 : G = H_0 \supseteq \cdots \supseteq H_n = \{e\}$
$\Sigma_2 : G = K_0 \supseteq \cdots \supseteq K_p = \{e\}$

Idea: ① For each $i = 0, \ldots, n-1$, use Σ_2 to insert $(p-1)$ many groups $\{H_{ij}'\}_{j=1}^{p-1}$ in between H_i & H_{i+1}
$\rightsquigarrow$ get Σ_1' finer than Σ_1

② Similarly, use Σ_1 to insert $(n-1)$ many subgroups $\{K_{j,i}'\}_{i=1}^{n-1}$ in between K_j & K_{j+1}.
$\rightsquigarrow$ get Σ_2' finer than Σ_2.

③ Show Σ_1' & Σ_2' are equivalent.

Define: $i = 0, \ldots, n-1$, $j = 0, \ldots, p-1$

$$\boxed{H_{ij}' := H_{i+1}(H_i \cap K_j)} \quad \& \quad \boxed{K_{j,i}' := K_{j+1}(H_i \cap K_j)}$$

It is clear that: $H_{i,0}' = H_i$, $H_{ip}' = H_{i+1}$; $K_{j,0}' = K_j$, $K_{j,n}' = K_{j+1}$.

- $H_{i,j+1}' < H_{i,j}'$; $K_{j,i+1}' < K_{j,i}'$ $\forall i, j$ are subgroups by 3rd Iso theorem ($H < G$, $N \triangleleft G \Rightarrow N \cdot H = H \cdot N < G$)

We need to check these subgroups are normal!

<u>Claim</u>: (i) $H'_{i,j+1} \lhd H'_{i,j}$, $K'_{j,i+1} \lhd K'_{j,i}$

(ii) $H'_{i,j}/H'_{i,j+1} \simeq K'_{j,i}/K'_{j,i+1}$

To simplify notation, write $H = H_i \rhd H' = H_{i+1}$

$K = K_j \rhd K' = K_{j+1}$

The claims will follow using Zassenhaus' Lemma. $\square$

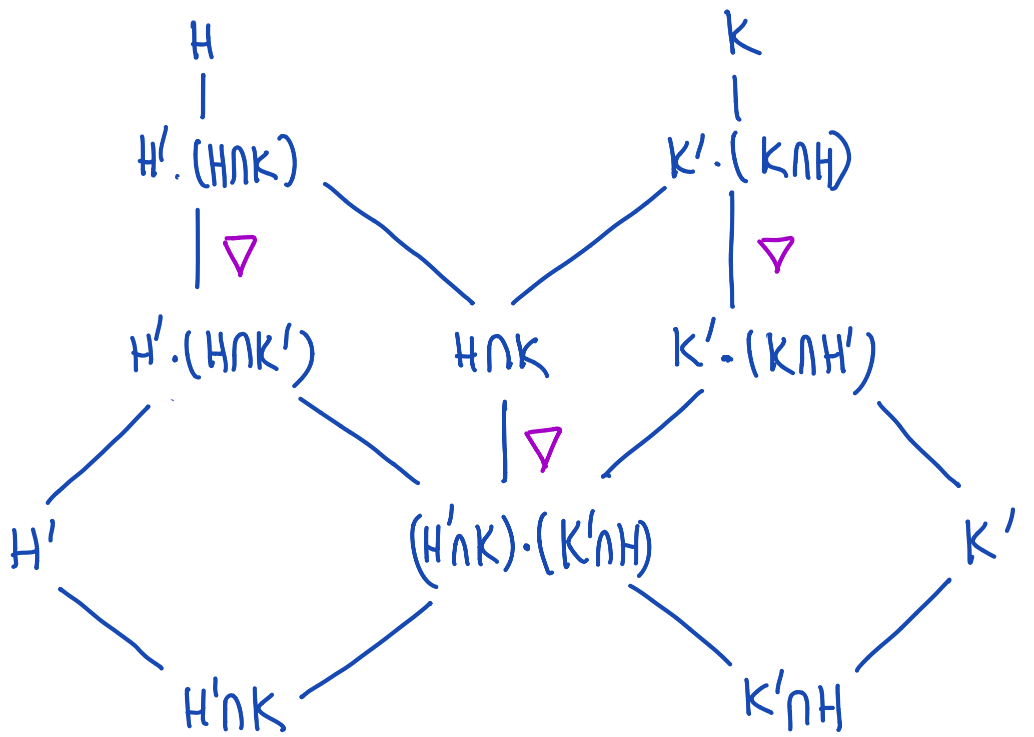
<u>Lemma (Zassenhaus)</u>. Fix a group G, H, K two subgroups of G & $H' \lhd H$, $K' \lhd K$. Then:

(i) $H' \cdot (H \cap K') \lhd H' \cdot (H \cap K)$

$K' \cdot (H' \cap K) \lhd K' \cdot (H \cap K)$

(ii) $\dfrac{H' \cdot (H \cap K)}{H' \cdot (H \cap K')} \simeq \dfrac{K' \cdot (H \cap K)}{K' \cdot (H' \cap K)}$

<u>Proof</u>: The next picture outlines the main steps of the proof:



<u>STEP 1</u> $(H' \cap K) \cdot (K' \cap H) \lhd H \cap K$

This is true because $H' \lhd H$ so $H' \cap K \lhd H \cap K$
$K' \lhd K$ so $K' \cap H \lhd H \cap K$ $\Big\} \Rightarrow$ By 3rd Iso

$(H' \cap K)(K' \cap H) < H \cap K$ (use $H' \cap K \lhd H \cap K$, $K' \cap H < H \cap K$.

But $g(H'\cap K)\, K'\cap H\, g^{-1} = g(H'\cap K)g^{-1} g(K'\cap H)g^{-1} \subseteq (H'\cap K)(K'\cap H)$ for $g\in H\cap K$

so $(H'\cap K)(K'\cap H) \triangleleft H\cap K$.

STEP 2: $H'(H\cap K') \triangleleft H'(H\cap K)$

This follows from a more general statement

Lemma: If $\mathcal{G}$ is a group, $G_1 < \mathcal{G}$, $N \triangleleft \mathcal{G}$ & $G_2 \triangleleft G_1$, then $N\cdot G_2 \triangleleft N\cdot G_1$

Pf/ $G_2 \cdot N = N\cdot G_2 < \mathcal{G}$ & $G_1 N = NG_1 < \mathcal{G}$ by 3rd Isomorphism Thm.

Then $x = g_1 n n' g_2 (g_1 n)^{-1} = g_1 n n' g_2 n^{-1} g_1^{-1} = \underbrace{(g_1 n g_1^{-1})}_{\in N}\underbrace{(g_1 n' g_1^{-1})}_{\in N}\underbrace{(g_1 g_2 g_1^{-1})}_{\in G_2}\underbrace{(g_1 n^{-1} g_1^{-1})}_{\in N}$

$\forall g_1\in G_1$, $n, n'\in N$ $g_2\in G_2$.

$\Rightarrow x \in N N G_2 N = N G_2 N = N N G_2 = NG_2$ so $G_2 N \triangleleft G_1 N$ $\square$

(*)

Then the claim follows by taking $\mathcal{G} = H$, $N = H'$, $G_1 = H\cap K$, $G_2 = H\cap K'$.

STEP 3: Use the 3rd Isomorphism Theorem:

$$\frac{H'(H\cap K)}{H'(H\cap K')} \simeq \frac{H\cap K}{(H\cap K)\cap (H'\cdot(H\cap K'))} \qquad \begin{cases} N = H'\cdot(H\cap K') \triangleleft H \\ \tilde{H} = H\cap K \end{cases}$$

Claim: $(H\cap K)\cap (H'\cdot(H\cap K')) = (H'\cap K)\cdot(K'\cap H)$

Pf/ $(H'\cap K)(K'\cap H) \subset (H\cap K)\cap(H'\cdot(H\cap K'))$ is clear

Conversely, let $x = a\cdot b \in H'(H\cap K')\cap(H\cap K)$ with $a\in H'$, $b\in \underset{H\cap K}{\overset{\cap}{H\cap K'}}$

$\Rightarrow a = x b^{-1} \in (H\cap K)\cdot(H\cap K) \subseteq H\cap K$

$\Rightarrow a \in H'\cap(H\cap K) = H'\cap K$.

Thus, $x = ab \in (H'\cap K)(H\cap K')$. $\square$

Swapping the roles of H & K, H' & K', combined with the Claim we get

$$\frac{K'(H\cap K)}{K'(H\cap K')} \simeq \frac{H\cap K}{(H\cap K')(K\cap H')} \simeq \frac{H'(H\cap K)}{H'(H\cap K')}$$ $\square$

(*) Alternative proof of the Lemma:

Let $\pi : \mathcal{G} \longrightarrow \mathcal{G}/N$ be the natural projection & $\overline{G}_1 = \pi(G_1)$

Restrict π to G_1 to get $\pi : G_1 \twoheadrightarrow \overline{G_1}$, hence $\overline{G_2} := \pi(G_2)$ is a normal subgroup of $\overline{G}_1$ (because π is surjective).

Now, consider the homomorphism defined by:

$$G_1 \cdot N \hookrightarrow \mathcal{G} \longrightarrow \mathcal{G}/N$$

$$\alpha : G_1 N \longrightarrow \overline{G_1}$$

Then $\alpha^{-1}(\overline{G_2}) = G_2 \cdot N$ is the inverse image of a normal subgroup, hence normal.