Lecture 12: Jordan-Hölder & Derived Series

Last time: Discussed composition series

- A <u>composition series</u> of a group G is a finite sequence of subgroups of G

$$\Sigma: \quad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = \{e\}$$

such that $G_{j+1} \triangleleft G_j$ is normal for all $j = 0, \dots, k-1$.

- Graded pieces: $gr_i(G) := G_i / G_{i+1}$ $\quad 0 \leq i \leq k-1$.
- Refinement: add terms to the composition series while remaining one
- Equivalence:
  - same number of terms
  - — graded pieces, counted with multiplicity (up to permutation)

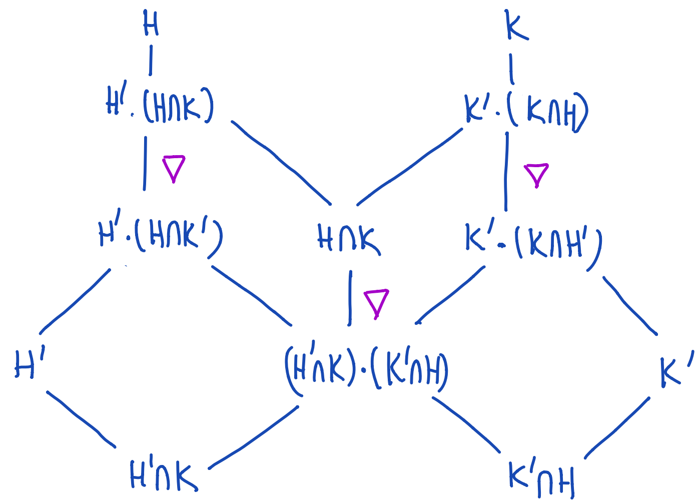
Theorem (Schrier) Any two composition series of a group G have a "common refinement", up to equivalence.

Lemma (Zassenhaus). Fix a group G, H, K two subgroups of G & $H' \triangleleft H$, $K' \triangleleft K$. Then:

(i) $H' \cdot (H \cap K') \triangleleft H' \cdot (H \cap K)$

$K' \cdot (H' \cap K) \triangleleft K' \cdot (H \cap K)$

(ii) $\dfrac{H' \cdot (H \cap K)}{H' \cdot (H \cap K')} \simeq \dfrac{H \cap K}{(H' \cap K)(K' \cap H)} \simeq \dfrac{K' \cdot (H \cap K)}{K' \cdot (H' \cap K)}$



TODAY:
- Discuss maximally refined comp. series = Jordan Hölder series.
- Special comp series build out of commutators = Derived series.

§1. Jordan-Hölder Series

Definition A composition series $\Sigma: G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$ is said to be a <u>Jordan-Hölder series</u> if:

(i) Σ is strictly decreasing (ie $G_{j+1} \subsetneq G_j$ $\forall j = 0, \dots, n-1$)

(ii) There is no strictly decreasing composition series distinct from Σ and finer than Σ.

<u>Proposition:</u> A composition series Σ of G is Jordan-Hölder (or JH for short) if and only if $gr_i^{\Sigma}(G)$ is simple for all $i=0,\dots,n-1$.

(<u>Recall</u>: $\{e\}$ is not simple; G is simple if $H \lhd G \Rightarrow H=\{e\}$ or G)

<u>Proof</u>: Note that a composition series is strictly decreasing if and only if none of its associated quotients is $\{e\}$.

Let $\Sigma: G=G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_n=\{e\}$ be a strictly decreasing composition series that is not JH. Then, there exists a strictly decreasing series Σ' finer than Σ. Thus, we can find $i=0,\dots,n-1$ where $G_{i+1} \subsetneq G_i$ are <u>not</u> consecutive in Σ'. That is, there exist intermediary normal subgroups:

$$G_{i+1} \underset{\neq}{\lhd} H_k \lhd \cdots H_2 \underset{\neq}{\lhd} H_1 \underset{\neq}{\lhd} G_i$$

In particular, $G_{i+1} \lhd H_1$ since $G_{i+1} \lhd G_i$ & $G_{i+1} < H_1 < G_i$.

Hence, H_1/G_{i+1} is a nontrivial normal subgroup of G_i/G_{i+1}, so $gr_i(G)$ is not simple.

Conversely, assume $\Sigma: G=G_0 \supsetneq \cdots \supsetneq G_n=\{e\}$ is a strictly decreasing composition series, one of whose graded pieces, say G_i/G_{i+1} is not simple. By the second Isomorphism Theorem, a proper, nontrivial normal subgroup of G_i/G_{i+1} is of the form H/G_{i+1} for some intermediate normal subgroup

$$G_{i+1} < H \lhd G_{i+1} \quad . \text{ Thus, } G_{i+1} \underset{\neq}{\lhd} H \underset{\neq}{\lhd} G_{i+1})$$

Conclude: $\Sigma': G=G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_i \supsetneq H \supsetneq G_{i+1} \supsetneq \cdots \supsetneq G_n=\{e\}$ is finer than Σ, so Σ is not J-H. $\square$

⚠ A general group G need <u>NOT</u> posses a JH series

<u>Ex.</u> $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq 8\mathbb{Z} \supsetneq \cdots \supsetneq G_k = 2^k\mathbb{Z} \supsetneq \cdots$ cannot terminate

However, every <u>finite</u> group G has a Jordan-Hölder (By induction on $|G|$. More precisely, pick H_1 maximal among all proper, normal subgroups of G, recursively let H_{n+1} be maximal among proper normal subgroups of H_n. This procedure must halt, at most $|G|$ steps later, thus forming a JH series)

<u>Theorem</u> (Jordan-Hölder) Two Jordan-Hölder series of a group G are equivalent.

<u>Proof</u>: Let Σ_1, Σ_2 be two JH series of G. By Schrier's Thm, we can refine them to Σ_1' & Σ_2' where Σ_1' & Σ_2' are equivalent.

As Σ_1 (and Σ_2) is JH, Σ_1' (and Σ_2') is either identical to Σ_1 (resp. Σ_2) or it is obtained from Σ_1 (resp. Σ_2) by repeating some terms. As the series of quotients of Σ_1' & Σ_2' differ only in the order of the graded pieces, after removing all trivial quotients, the same is true for Σ_1 & Σ_2 $\square$

<u>Ex</u>: $G = \mathbb{Z}/6\mathbb{Z}$

$\Sigma_1 : \mathbb{Z}/6\mathbb{Z} \supsetneq \mathbb{Z}/3\mathbb{Z} \supsetneq \{e\}$ JH

$\Sigma_2 : \mathbb{Z}/6\mathbb{Z} \supsetneq \mathbb{Z}/2\mathbb{Z} \supsetneq \{e\}$ JH

graded pieces:

$$gr_0^{\Sigma_1}(G) = \mathbb{Z}/2\mathbb{Z} = gr_1^{\Sigma_2}(G)$$
$$gr_1^{\Sigma_1}(G) = \mathbb{Z}/3\mathbb{Z} = gr_0^{\Sigma_2}(G)$$

<u>Corollary</u>: Let G be a group that admits a JH series. If Σ is any strictly decreasing composition series of G, then there exists a JH series refining Σ.

<u>Sketch of a proof</u>: Let Σ_0 be a J-H series of G. By Schrier's Thm, we can find Σ_0' & Σ' two equivalent composition series refining Σ_0 & Σ, resp. The proof of JH Theorem ensures that Σ_0' is JH & so Σ' is also JH.

<u>Example 1</u>: $G = \mathbb{Z}/p^k\mathbb{Z} = \langle g \rangle$ $\quad k > 1$ $\quad$ graded pieces for JH $= \mathbb{Z}/p\mathbb{Z}$. (simple & order $|p^k$)

$$\Sigma: G = G_0 \supseteq G_1 = \underset{\langle g^p \rangle}{\mathbb{Z}/p^{k-1}\mathbb{Z}} \supseteq G_2 = \underset{\langle g^{p^2} \rangle}{\mathbb{Z}/p^{k-2}\mathbb{Z}} \supseteq \cdots \supseteq G_{k-1} = \underset{\langle g^{p^{k-1}} \rangle}{\mathbb{Z}/p\mathbb{Z}} \supseteq G_k = \{e\}$$

is JH.

Example 2: $G = \mathbb{Z}/n\mathbb{Z}$ How to build a JH series for G?

- If n is prime, G is simple so $G \geq \{e\}$ is JH
- If n is not prime, write $n = p_1^{a_1} \cdots p_r^{a_r}$ prime decomposition.

$$\Rightarrow G = \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/\tfrac{n}{p_1^{a_1}}\mathbb{Z} = \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \left(\mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \mathbb{Z}/\tfrac{n}{p_1^{a_1}p_2^{a_2}}\mathbb{Z} \right) = \cdots$$

$$= \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r^{a_r}\mathbb{Z}$$

$$\Rightarrow G = G_0 \gneq G_1 := \mathbb{Z}/\tfrac{n}{p_1^{a_1}}\mathbb{Z} \geq G_2 := \mathbb{Z}/\tfrac{n}{p_1^{a_1}p_2^{a_2}}\mathbb{Z} \geq \cdots \geq G_{r-1} := \mathbb{Z}/p_r^{a_r}\mathbb{Z} \geq G_r = \{e\}$$

comp series with graded pieces $= p$-groups.

We can refine each $G_i = \mathbb{Z}/\tfrac{n}{p_1^{a_1}\cdots p_i^{a_i}}\mathbb{Z} \geq G_{i+1} = \mathbb{Z}/\tfrac{n}{p_1^{a_1}\cdots p_{i+1}^{a_{i+1}}}\mathbb{Z}$

by lifting a JH series of $\mathbb{Z}/p_{i+1}^{a_{i+1}}\mathbb{Z}$ (use Example 1)

§2 Derived Series of a group

Recall: $[G:G] = \langle \underbrace{aba^{-1}b^{-1}}_{=:[a:b]} : a, b \in G \rangle$ commutator subgroup of G.

Definition: given $A, B < G$, we consider
$(A:B) = \langle aba^{-1}b^{-1} : a \in A, b \in B \rangle$

Lemma: If $A, B \lhd G$, then $(A,B) \lhd G$.

Proof: For all $g \in G$, $a \in A$, $b \in B$:

$$g\,aba^{-1}b^{-1}g^{-1} = \underbrace{(gag^{-1})}_{\in A}\underbrace{(gbg^{-1})}_{\in B}(gag^{-1})^{-1}(gbg^{-1})^{-1} = [\overset{\in A}{gag^{-1}} : \overset{\in B}{gbg^{-1}}] \in (A,B) \quad \square$$

We will use commutators to define a composition series for G in a recursive way:

Corollary: We define recursively:

$$D^0(G) = G \quad , \quad D^{n+1}(G) = D(D^n(G)) := (D^n(G), D^n(G))$$

Then, each $D^n(G)$ is normal in G, and $D^n(G)/D^{n+1}(G)$ is abelian (by Problem 11 HW1)

Definition: The sequence $\Sigma: G = D^0(G) \geq D^1(G) \geq \cdots$ is called the <u>derived series</u> of G.

Q: When is Σ a composition series? **A:** Need $D^N(G) = \{e\}$ for some N.

Definition: We say G is <u>solvable</u> if there exists $N \geq 0$ with $D^N(G) = \{e\}$

Remarks: ① The term "solvable" originates from Galois Theory (Math 6112)

② $D^0(G) = \{e\} \iff G$ is trivial.

③ $D^1(G) = \{e\} \iff G$ is abelian (hence, all abelian groups are solvable)

Proposition: If G is non-abelian & simple, then $D^n(G) = G$ for all $n \geq 0$, so G is not solvable.

Pf/ $\left.\begin{array}{l} D(G) \neq \{e\} \text{ (otherwise, } G \text{ would be abelian),} \\ D(G) \triangleleft G \text{ normal} \implies D(G) = G. \end{array}\right\} \Rightarrow D^n(G) = D^0(G) = G$ for all n.

Example: $G = D_n$, $D_n^0 = \langle \rho^2 \rangle$ which is abelian.

Pf/ • $[s, \rho] = s\rho s^{-1}\rho^{-1} = s\rho s\rho^{-1} = \rho^{-2}$

• $[s\rho^i : \rho^j] = s\rho^i\rho^j(s\rho^i)^{-1}\rho^{-j} = s\rho^{i+j}\rho^{-i}s\rho^{-j} = \rho^{-2j}$

• $[s\rho^i : s\rho^j] = s\rho^i s\rho^j (s\rho^i)^{-1}(s\rho^j)^{-1} = \rho^{j-i}\rho^{-i}s\rho^{-j}s$

$= \rho^{j-2i}\rho^{j} = \rho^{2(j-i)}$.

• $[\rho^i : \rho^j] = e$

So $\mathcal{D}: G = D_n \supseteq D^0(G) = \langle \rho^2 \rangle \geq D^2(G) = \{e\}$ & $\boxed{D_n \text{ is solvable}}$

• By our earlier discussion, we can find a JH series refining the desired series $\mathcal{D}$:

$$gr_0^{\mathcal{D}}(D_n) = \frac{\langle \rho, s\rangle}{\langle \rho^2\rangle} \qquad gr_1^{\mathcal{D}}(D_n) = \frac{\langle \rho^2\rangle}{\langle e\rangle}$$

$\Rightarrow$ The answer depends on the parity of n!

If n is odd: $\langle \rho^2\rangle = \langle \rho\rangle \simeq \mathbb{Z}/n\mathbb{Z}$

$gr_0^{\mathcal{D}}(D_n) = \frac{\langle \rho, s\rangle}{\langle \rho\rangle} \cong \mathbb{Z}/2\mathbb{Z}$ (order is $\frac{2n}{n} = 2$) simple

$gr_1^{\mathcal{D}}(D_n) = \langle \rho\rangle \simeq \mathbb{Z}/n\mathbb{Z}$ not simple if n is not prime

$\rightsquigarrow$ We refine $\langle \rho\rangle = \mathbb{Z}/n\mathbb{Z} \supseteq \langle e\rangle$ to a JH-series using Example 2.

If n is even: $\langle \rho^2\rangle \simeq \mathbb{Z}/\frac{n}{2}\mathbb{Z}$ $\qquad n = 2m$

• $|gr_0^{\mathcal{D}}(D_n)| = \frac{2n}{m} = 4 = 2^2 \Rightarrow gr_0(D_n) \simeq \mathbb{Z}/4\mathbb{Z}$ or $\boxed{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$

Classification of p^2-order gps.

$\langle \bar\rho\rangle \times \langle \bar s\rangle$

• $gr_1^{\mathcal{D}}(D_n) \simeq \mathbb{Z}/m\mathbb{Z}$ $\rightsquigarrow$ can be refined to JH series.

We refine $\langle s, \rho\rangle \supseteq \langle \rho^2\rangle$ by $\langle s, \rho\rangle \supseteq \langle \rho\rangle \supseteq \langle \rho^2\rangle$

$\frac{\langle s, \rho\rangle}{\langle \rho\rangle} \simeq \langle s\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ simple

$\frac{\langle \rho\rangle}{\langle \rho^2\rangle} \simeq \mathbb{Z}/2\mathbb{Z}$ simple.

Refining $\langle \rho^2\rangle \supseteq \langle e\rangle$ using a JH for $\mathbb{Z}/m\mathbb{Z}$ gives a JH series for D_{2m}.