Lecture 13: Solvable & Nilpotent Groups

Recall: We defined the derived series of a group G via

$(A:B) = \langle [a,b] = aba^{-1}b^{-1} : a\in A, b\in B\rangle \trianglelefteq G$ if $A, B \trianglelefteq G$

- $D^0(G) = G$; $D^{n+1}(G) = (D^n(G), D^n(G))$
- Main property: $H/(H,H)$ is abelian, hence any subgroup of H containing (H,H) is normal. Conversely, if $A\trianglelefteq H$ & H/A is abelian, then $(H,H)\subseteq A$.

§1. Solvable groups:

- Solvable groups: $D^N(G) = \{e\}$ for some $N\geq 0$.

Equiv: $\mathcal{D}$: $G \supseteq D(G) \supseteq D^2(G) \supseteq \ldots\ldots \supseteq D^N(G) = \{e\}$ is a composition series for G & $D^j(G)/D^{j+1}(G)$ is abelian $\forall j$

Lemma: The group of upper triangular invertible matrices is solvable.

Eg:
$$B := \left\{\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} \quad a,d\in\mathbb{C}^\times, b\in\mathbb{C}\right\}$$
$$\cup$$
$$D(B) = \left\{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix} : x\in\mathbb{C}\right\}$$
$$\cup$$
$$D^2(B) = \{e\}$$

Pf/ • Assume $D(B)$ is as claimed, then $D(B)\simeq\mathbb{C}$ abelian
$\Rightarrow D^2(B) = \{e\}$.

- We argue the claim for $D(B)$ holds by explicit computation

$A = \begin{pmatrix} a & b\\ 0 & d\end{pmatrix}$, $A' = \begin{pmatrix} a' & b'\\ 0 & d'\end{pmatrix}$

$$AA'A^{-1}(A')^{-1} = \frac{1}{ad\,a'd'}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\begin{pmatrix} a' & b'\\ 0 & d'\end{pmatrix}\begin{pmatrix} d & -b\\ 0 & a\end{pmatrix}\begin{pmatrix} d' & -b'\\ 0 & a'\end{pmatrix}$$

$$= \frac{1}{ada'd'} \begin{pmatrix} aa' & b'a+bd' \\ 0 & dd' \end{pmatrix} \begin{pmatrix} dd' & -b'd-ba' \\ 0 & aa' \end{pmatrix}$$

$$= \frac{1}{ada'd'} \begin{pmatrix} aa'dd' & aa'(b'a+bd'-b'd-ba') \\ 0 & dd'aa' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad x = \frac{1}{dd'}(b'(a-d) - b(a'-d'))$$

$\Rightarrow D(B) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \; x \in \mathbb{C} \right\} \simeq \mathbb{C}$ abelian

Recall: ① $D^0(G) = \{e\} \iff G$ is trivial
② $D^1(G) = \{e\} \iff G$ is abelian
} Both examples of solvable gps

③ If G is non-abelian & simple, then $D^n(G) = G$ for all $n \geq 0$, so G is not solvable.

Ex: $D^0(S_n) = A_n$ and A_n is simple (nonabelian) for $n \geq 0$, so S_n is not solvable for $n \geq 5$.

Pf/ Assume A_n is simple for $n \geq 5$. Then

. $[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1} \in A_n$ for all $\sigma, \tau \Rightarrow D^0(S_n) \subseteq A_n$

. Since S_n is not abelian $D^0(S_n) \neq \{e\}$

. $D^0(S_n) \lhd S_n \Rightarrow \underset{\neq \{e\}}{D^0(S_n)} \lhd A_n$ forces $D^0(S_n) = A_n$.

Obs: This will be used to show that quintic or higher degree polynomials cannot be solved by radicals (like the quadratic polynomials in $\mathbb{C}[x]$)

Theorem 1: Let G be a group, and assume it has a composition series

$$\Sigma: G = G_0 \geq G_1 \geq \cdots \geq G_n = \{e\}$$

where each $gr_i(G) = G_i / G_{i+1}$ is abelian. Then, G is solvable.

Proof: We will show $D^j(G) \subseteq G_j$ $\forall j$ by induction on j. In particular $D^n(G) \subseteq \{e\}$ so $D^n(G) = \{e\}$ & G is solvable.

Base case: $j=0$ is clear since $D^0(G) = G = G_0$.

Inductive Step: Assume $D^j(G) \subset G_j$. Since G_j/G_{j+1} is abelian, then $D(G_j) \subseteq G_{j+1}$. But $D(G_j) = D^{j+1}(G)$

So $D^{j+1}(G) = D(D(G_j)) \subseteq D(G_j) \subseteq G_{j+1}$, as we wanted $\square$

Corollary: G is solvable $\Leftrightarrow$ $\exists$ comp series with abelian graded pieces.

Theorem 2: Let G be a p-group. Then, there exists a comp series for G

$$G = G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \cdots \supsetneq G_r = \{e\}$$

with $gr_j(G) = G_j/G_{j+1} \simeq \mathbb{Z}/p\mathbb{Z}$ $\forall i = 0, \ldots, r-1$

Proof: By induction on k where $|G| = p^k$.

- Base case: $k=1$ is clear since $G \simeq \mathbb{Z}/p\mathbb{Z}$ ($r=1$ will do)
- Inductive Step: Fix $\mathcal{Z} = Z(G)$ center of G. Recall that the center of a p-group is always nontrivial.

  Pick $x \in \mathcal{Z} \setminus \{e\}$. So $\operatorname{order}(x) = p^s$ & $s \geq 1$ & $x^{p^{s-1}}$ has order p. Set $H = \langle x^{p^{s-1}} \rangle \subset \mathcal{Z}$ so $H \trianglelefteq G$ & $H \simeq \mathbb{Z}/p\mathbb{Z}$

  Now $|G/H| = p^{k-1}$ & by inductive hypothesis we can find a composition series for G/H:

$$G/H = \overline{G_0} \supseteq \overline{G_1} \supseteq \cdots \supseteq \overline{G_r} = \{eH\}$$

with $\overline{G_i}/\overline{G_{i+1}} \simeq \mathbb{Z}/p\mathbb{Z}$ for all $i = 0, \ldots, r-1$.

Now, we use $\pi : G \twoheadrightarrow G/H$ to define $G_j = \pi^{-1}(\overline{G_j})$.

By 2^{nd} Isomorphism Theorem $G_{j+1} \lhd G_j$ $\forall j$ & $H \leq G_j$ $\forall j$

Furthermore
- $G_j / G_{j+1} \simeq \dfrac{G_j/H}{G_{j+1}/H} = \overline{G_j} / \overline{G_{j+1}} \simeq \mathbb{Z}/p\mathbb{Z}$
- $G_r = H \simeq \mathbb{Z}/p\mathbb{Z}$; $G_0 = G$

So $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_r = H \supseteq G_{r+1} = \{e\}$ is the composition series we were after. $\square$

Corollary 2: Every p-group is solvable.

Pf/ Take the composition series from Thm 2 & use $\mathbb{Z}/p\mathbb{Z}$ is abelian $\square$

Proposition: Let G be a group & $N \lhd G$. Then, G is solvable if, and only if, N & G/N are.

Equivalent statement: $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$ ses

Then G_2 is solvable $\iff$ G_1 & G_3 are solvable.

Example $G = D_n$, $N = \langle \rho \rangle \lhd G$ $G/N \simeq \mathbb{Z}/2\mathbb{Z}$ both solvable (abelian),

so D_n is solvable.

Proof: ($\Rightarrow$) First, assume G is solvable & pick $n \geq 0$ with $D^n(G) = \{e\}$

Then $D^n(N) \subseteq D^n(G) = \{e\}$ $\Rightarrow$ N is solvable.

$\hookrightarrow$ clear by easy induction $D^j(N) \subseteq D^j(G)$ for all j

If $\pi : G \twoheadrightarrow G/N$ is the natural projection, then:

$$\pi(D(G)) = \pi((G:G)) \underset{\pi \text{ group hom}}{=} (\pi(G); \pi(G)) = D(G/N)$$

Thus $\pi(D^{j+1}(G)) = D(D^j(G/N)) = D^{j+1}(G/N)$

So $D^n(G/N) = \pi(\{e\}) = \{e_{G/N}\}$ so G/N is solvable

($\Leftarrow$) Now, assume N & G/N are solvable. By Theorem 1 we have composition series for N & G/N with abelian graded pieces

Σ: $N = N_0 \geq N_1 \geq \cdots \geq N_k = \{e\}$ $\quad \frac{N_i}{N_{i+1}}$ abelian $\forall i = 0, \ldots, k-1$.

Σ': $G/N = \overline{G_0} \geq \overline{G_1} \geq \cdots \geq \overline{G_s} = \{e_{G/N}\}$ $\quad \overline{G_j}/\overline{G_{j+1}}$ —— $\forall j = 0, \ldots, s-1$.

Set $\pi : G \twoheadrightarrow G/N$ & $\boxed{G_j := \pi^{-1}(\overline{G_j})}$ $\forall j = 0, \ldots, s$

So $G_s = N$, $G_0 = G$ & $G_j/G_{j+1} \simeq \overline{G_j}/\overline{G_{j+1}}$ abelian $(N < G_j \ \forall j)$

Set $\boxed{G_{s+i} := N_i}$ for $i = 1, \ldots, k$. Then:

Σ'' $G = G_0 \geq G_1 \geq \cdots \geq G_s = N \geq G_{s+1} \geq \cdots \geq G_{s+k} = \{e\}$

is a comp series for G with abelian graded pieces. By Thm 1, G is solvable. $\square$

Q: What can we say about Jordan-Hölder series of finite, solvable gps?

Proposition: Fix G a finite group. Then, the following are equivalent:

(1) G is solvable

(2) $gr_j^{\Sigma}(G)$ is cyclic of prime order $\forall j$ for some Jordan-Hölder series Σ of G.

(3) $gr_j^{\Sigma}(G)$ is cyclic of prime order $\forall j$ for ALL Jordan-Hölder series Σ of G.

Proof: (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (3) Assume G is solvable & pick any Jordan-Hölder series Σ_1 of G (it exists because G is finite)

Pick a comp series Σ_2 of G will abelian graded pieces (it exists because G is solvable). By Schrier's Thm we can find refinement of Σ_1 & Σ_2 that are equivalent. Call them Σ_1' & Σ_2', respectively

- So the graded pieces of Σ_1' are either trivial or simple
- The graded pieces of Σ_2' are abelian (since we are refining Σ_1 & its graded pieces were abelian)

By equivalence, the graded pieces of Σ_1' are trivial or simple & abelian.

$\Rightarrow gr_i^{\Sigma_1'}(G) = \{e\}$ or $\mathbb{Z}/p\mathbb{Z}$ with p prime.

Since Σ_1 was JH to begin with, we conclude that $gr_i^{\Sigma_1}(G) \simeq \mathbb{Z}/p_i\mathbb{Z}$ where $p_i > 0$ is prime for all i. Thus (3) holds. □

§2. Lower Central Series

We now define a new sequence involving a new commutator.

Set $C^1(G) = G$

$$C^{n+1}(G) = (G, C^n(G)) \quad \forall n \geq 1 \quad (\triangleleft G \text{ if } C^n(G) \triangleleft G)$$

By induction on n we see $C^n(G) \triangleleft G \ \forall n$

Lemma: $C^{n+1}(G) < C^n(G)$ so $C^{n+1}(G) \triangleleft C^n(G)$

Pf/ $C^{n+1}(G) = \langle \underbrace{gxg^{-1}}_{\in C^n(G)} \underset{C^n(G)}{x^{-1}} : g \in G, x \in C^n(G) \rangle < C^n(G)$

But $C^n(G) \triangleleft G$

Since $C^{n+1}(G) \triangleleft G$, we conclude: $C^{n+1}(G) \triangleleft C^n(G)$. □

We build the sequence:

$$\mathscr{C}: \quad G = C^1(G) \geq C^2(G) \geq C^3(G) \geq \ldots$$

Definition: G is nilpotent if $\exists\, n \geq 1$ such that $C^n(G) = \{e\}$.

Equivalently, $\mathscr{C}$ is a composition series for G.