Lecture 19: Ring of fractions, modules of fractions

Last Time: We defined multiplicatively closed sets S & the ring of fractions $S^{-1}R$, where R is commutative.

§1. Ring of fractions.

Def: Fix a commutative ring R & $S \subset R$. We say S is a multiplicatively closed set if:

(i) $0 \notin S$ ⇝ otherwise localization gives a set with $0=1$.

(ii) $1 \in S$

(iii) $a, b \in S \Rightarrow ab \in S$.

- Equivalence relation $\sim$ on $R \times S$:

$$(a,s) \sim (b,t) \iff \exists\, s' \in S \text{ with } s'(at-bs)=0$$

Def The ring of fractions of R relative to S, denoted by $S^{-1}R$ is the set $R \times S/\sim$ with

1. Addition: $(a,s)+(b,t) = (at+bs, st)$
2. Multiplication: $(a,s)\cdot(b,t) = (ab, st)$
3. Neutral elements: $0 = (0,1)$ & $1 = (1,1)$

(Think of (a,s) in $R^{-1}S$ as $\frac{a}{s}$.)

Special case: R is an integral domain (no zero divisors)

Then $S = R \setminus \{0\}$ is a multiplicatively closed set.

Then $S^{-1}R$ is a field called the field of fractions

Sometimes denoted by $\mathrm{Quot}(R)$

$(a,s) \neq (0,1)$ is invertible & $(a,s)^{-1} = (s,a)$.

$(\iff a \neq 0)$

Ex: ① $R=\mathbb{Z}$, $\mathrm{Quot}(R)=\mathbb{Q}$ $(a,b) \longleftrightarrow \frac{a}{b}$

$(a,b)\sim(c,d) \iff \exists\, s\in\mathbb{Z}\smallsetminus\{0\}$ with $s(ad-bc)=0$

But $\mathbb{Z}$ is a domain, so $ad-bc=0$ (since $s\neq 0$)

If $m=\gcd(a,b)$, $n=\gcd(d,c)$ then $\frac{a}{m}\frac{d}{n}=\frac{b}{m}\frac{c}{n}$ forces

$\frac{a}{m}=k\frac{c}{n}$ & $\frac{d}{n}=k\frac{b}{m}$ for $k\in\mathbb{Z}$

In conclusion: $\frac{a}{b}=\frac{a/m}{b/m}=\frac{c/n}{d/n}=\frac{c}{d}$.

② $R=\mathbb{Z}[x]$, $\mathrm{Quot}(R)=\mathbb{Q}(x)=\left\{\frac{P(x)}{Q(x)} : P,Q\in\mathbb{Q}[x],\ Q\neq 0\right\}$

Another special example of S:

- R commutative $S=$ set of nonzero divisors of R

Thm: S is multiplicatively closed

Def: $S^{-1}R=$ Total ring of fractions $=\mathrm{Quot}(R)$

- From polynomials to Laurent polynomials:

$S=\{1, x^i : i\geq 1\}\in \mathbb{K}[x]$ is multiplicatively closed.

Thm: $S^{-1}\mathbb{K}[x]=\mathbb{K}[x,x^{-1}]$.

$(a,s)\sim(b,t) \iff x^n(at-bs)=0$ $n\geq 0$

Again, $at-bs=0$, $s,t\in S$ so $s=x^k$, $t=x^l$

So $a=b\,x^{k-l}\in\mathbb{K}[x]$ if $k\geq l$

$b=a\,x^{l-k}\in\mathbb{K}[x]$ if $l\geq k$

$\Longrightarrow \frac{a}{s}\in\mathbb{K}[x,x^{-1}]$.

· A degenerate example:

$R = \mathbb{Z}/6\mathbb{Z} \supset S = \{1,2,4\}$ mult. closed

In $S^{-1}R$ $\frac{r}{s} = 0 \; (=\frac{0}{1}) \iff \exists t \in S$ with $t(r\cdot 1 - s\cdot 0) = 0$
$tr = 0$

So $\frac{3}{s} = 0 \quad \forall s \in S \quad (2\cdot 3 = 0)$

$\frac{1}{s} \neq 0 \;, \; \frac{2}{s} \neq 0 \;, \; \frac{4}{s} \neq 0 \;, \; \frac{5}{s} \neq 0$

Claim: $S^{-1}R = \{0, 1, \frac{1}{2}\}$ (HW7)

§2 Universal Properties:

Fix R commutative ring & $S \subset R$ multiplicatively closed

Proposition: We have a natural ring homomorphism

$$j_S : R \longrightarrow S^{-1}R$$
$$a \longmapsto \frac{a}{1} \quad (= \text{class of } (a,1))$$

such that for every $t \in S$, $j_S(t)$ is invertible in $S^{-1}R$ (its inverse is $\frac{1}{t}$)

Proof: Definition of the ring structure on $S^{-1}R$ makes j_S a ring homomorphism.

- $\left(\frac{t}{1}\right)^{-1} = \frac{1}{t}$ because $(t,1)\cdot(1,t) = (t,t) = (1,1)$

Theorem: Fix B another commutative ring & let $f: R \longrightarrow B$ be a ring homomorphism such that $\forall t \in S$: $f(t) \in B$ is invertible. Then, there exists a unique ring homomorphism $\tilde{f}: S^{-1}R \longrightarrow B$ making $\tilde{f} \circ j_S = f$

Proof Want to show
$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "j_S"'] & B \\ S^{-1}R \arrow[ur, dashed, "\exists! \tilde{f}"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

st $\boxed{\tilde{f}\left(\frac{a}{s}\right) := f(a)\, f(s)^{-1}}$

- Well-defined: $\frac{a}{s} = \frac{b}{t} \iff \exists\, s' \in S$ with $s'(at - bs) = 0$

Then $\underset{B^*}{\underset{\cap}{f(s')}}\,\big(f(a)f(t) - f(b)f(s)\big) = 0$

$\implies f(a)\, f(t) = f(b) f(s)$

So $f(a)\, f(s)^{-1} = f(b)\, f(t)^{-1}$ in B

- Ring homomorphism:
  - $\tilde{f}\left(\frac{a}{s} + \frac{b}{t}\right) = \tilde{f}\left(\frac{at+bs}{st}\right) = f(at+bs)\, f(st)^{-1}$
  $= \big(f(a)f(t) + f(b)f(s)\big)\, f(s)^{-1} f(t)^{-1}$
  $= f(a)\, f(s)^{-1} + f(b)\, f(t)^{-1} = \tilde{f}\left(\frac{a}{s}\right) + \tilde{f}\left(\frac{b}{t}\right)$
  - $\tilde{f}\left(\frac{a}{s}\frac{b}{t}\right) = \tilde{f}\left(\frac{ab}{st}\right) = f(ab)\, f(st)^{-1}$
  $= f(a) f(b)\, f(s)^{-1} f(t)^{-1} = f(a) f(s)^{-1} f(b) f(t)^{-1}$
  $= \tilde{f}\left(\frac{a}{s}\right) \tilde{f}\left(\frac{b}{t}\right).$
  - $\tilde{f}(1) = \tilde{f}\left(\frac{1}{1}\right) = f(1)\, f(1)^{-1} = 1 \cdot 1^{-1} = 1.$
  - $\tilde{f} \circ j_S(a) = \tilde{f}\left(\frac{a}{1}\right) = f(a)\, f(1)^{-1} = f(a)\, 1^{-1} = f(a).$ $\square$

Lemma: $\operatorname{Ker}(j_S) = \{ a \in R : \exists\, s \in S \text{ with } sa = 0 \}$

Pf/ $j_S(a) = \frac{a}{1} = \frac{0}{1} \iff \exists\, s \in S$ st $s(a \cdot 1 - 0 \cdot 1) = 0$, ie $sa = 0$ $\square$

§3. Modules of fractions:

Fix R commutative ring & $S \subset R$ multiplicatively closed. Given M an R-module, we want to define a "module of fractions". Since R is an R-module, we can mimic the construction of $S^{-1}R = R \times S/\sim$.

- Equivalence Relation on $M \times S$:

  $(m,s) \sim (m',s')$ if there exists $t \in S$ such that:

  $$\underset{R}{t}(\underbrace{s'm - sm'}_{\text{in } M}) = 0 \text{ in } M.$$

Exercise: This is an equivalence relation (HW7)

Def: $S^{-1}M = \overline{M \times S}/\sim$ = module of fractions of M relative to S

Write $\overline{(m,s)}$ in $S^{-1}M$ as $\frac{m}{s}$.

We make this into an abelian group:

- Addition: $\frac{m}{s} + \frac{m'}{s'} = \frac{s'm + s\cdot m'}{ss'}$
- Neutral element: $\frac{0}{1}$

Exercise: Show this is well-defined & $-\left(\frac{m}{s}\right) = \frac{-m}{s}$.

Proposition: The abelian group $S^{-1}M$ is both an R-module & an $S^{-1}R$-module via

$$\frac{a}{s} \cdot \frac{m}{t} = \frac{a \cdot m}{st} \qquad \forall\, a \in R,\ s,t \in S\ m \in M$$

(R-module structure, use j_S: $a \cdot \frac{m}{t} = \frac{a}{1} \cdot \frac{m}{t}$)

__Exercise__: Show this is well-defined & satisfies the properties defining modules.

. We have an R-linear map $i_S : M \longrightarrow S^{-1}M$
$$m \longmapsto \frac{m}{1}$$

__Universal Property__: Write $\tilde{M} = S^{-1}M$

① For each $s \in S$, consider the gp hom $\rho_{(s)} : \tilde{M} \longrightarrow \tilde{M}$
$$m \longmapsto s \cdot m$$

We have $\rho_{(s)} \in \mathrm{End}_R(\tilde{M})$ & $\rho(s)$ is invertible
$$\rho(s)^{-1} = \rho\left(\tfrac{1}{s}\right) \in \mathrm{End}_R(\tilde{M})$$

② Let N be another module over R such that $\forall s \in S$
$\rho(s) : N \longrightarrow N$ is invertible (as $\rho_{(s)} \in \mathrm{End}_R(N)$)
$$n \longmapsto sn$$

Given an R-linear map $f : M \longrightarrow N$ there exists a unique R-linear map $\tilde{f} : \tilde{M} \longrightarrow N$ st

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_S"'] & N \\ \tilde{M} \arrow[ur, dashed, "\exists ! \tilde{f}"'] & \end{tikzcd}$$

commutes

__Proof__: $\tilde{f}\left(\frac{m}{s}\right) = \rho_{(s)}^{-1}(f(m))$

Need to show it is well-defined, R-linear (exercise)

$$\tilde{f} \circ i_S(m) = \tilde{f}\left(\frac{m}{1}\right) = \underbrace{\rho_{(1)}^{-1}}_{=\mathrm{id}}(f(m)) = f(m)$$

□

§4. Ideals in $S^{-1}R$:

Fix R commutative ring, $S \subset R$ mult. closed & $j_S : R \to S^{-1}R$ the natural ring homomorphism.

Given $\mathfrak{a} \subset R$ ideal, we define:

$$\boxed{S^{-1}\mathfrak{a}} = \text{ideal in } S^{-1}R \text{ generated by } j_S(\mathfrak{a}).$$

Proposition: $S^{-1}\mathfrak{a}$ agrees with the module of fractions of $\mathfrak{a}$ (viewed as an R-module) relative to S.

Theorem: Every ideal in $S^{-1}R$ is of this form ($= S^{-1}\mathfrak{a}$ for some $\mathfrak{a} \subset R$ ideal)

Furthermore, $S^{-1}\mathfrak{a} = S^{-1}R \iff S \cap \mathfrak{a} \neq \emptyset$.

Proof: Use $j_S : R \longrightarrow S^{-1}R$

Let $\mathfrak{b} \subset S^{-1}R$ be an ideal. & set $\mathfrak{a} = j_S^{-1}(\mathfrak{b})$.

- We know $\mathfrak{a}$ is an ideal because j_S is a ring homomorphism
- Claim: $S^{-1}\mathfrak{a} = \mathfrak{b}$

Pf/ $(\subseteq)$ $a \in \mathfrak{a}$, $\frac{r}{s} \in S^{-1}R \Rightarrow \frac{r \cdot a}{s \cdot 1} = \underbrace{\boxed{\frac{r}{s}}}_{\in S^{-1}R} \overbrace{\boxed{\frac{a}{1}}}^{\in \mathfrak{b}} \in \mathfrak{b}$ (ideal)

So $S^{-1}\mathfrak{a} = S^{-1}R\left(\frac{a}{1} : a \in \mathfrak{a}\right) \subseteq \mathfrak{b}$.

$(\supseteq)$ Pick $x \in \mathfrak{b}$. Then $x = \frac{y}{s}$ for some $y \in R$, $s \in S$

$\Rightarrow \frac{y}{1} = \underbrace{\boxed{\frac{s}{1}}}_{\in R} \cdot \underbrace{\boxed{\frac{y}{s}}}_{\in \mathfrak{b}}$ so $y \in \mathfrak{a}$ so $\mathfrak{b} \subseteq S^{-1}\mathfrak{a}$.

For the last part:

If $s \in S \cap \mathfrak{a}$, write $1 = \frac{s}{s} \in S^{-1}\mathfrak{a}$.

Conversely, assume $1 \in S^{-1}\mathfrak{a}$. Then, $\exists$ $a_1, \dots, a_n \in \mathfrak{a}$ & $\frac{r_1}{s_1}, \dots, \frac{r_n}{s_n} \in S^{-1}R$ with $1 = \sum_{i=1}^{n} \frac{r_i}{s_i} \cdot a_i = \sum_{i=1}^{n} \frac{(r_i a_i)}{s_i} = \sum_{i=1}^{n} \frac{b_i}{s}$

where $s = \prod_{i=1}^{n} s_i$ & $b_i = r_i a_i \prod_{j \neq i} s_j \in \mathfrak{a}$ $\forall i = 1, \dots, n$

BUT $\sum_{i=1}^{n} \frac{b_i}{s} = \frac{\sum_{i=1}^{n} b_i}{s} = \frac{b}{s}$ with $b = \sum_{i=1}^{n} b_i \in \mathfrak{a}$

So $1 = \frac{b}{s}$ for $b \in \mathfrak{a}$, $s \in S$ $\iff$ $\exists t \in S$ with

$0 = t(s \cdot 1 - b \cdot 1) = ts - tb$

So $\underbrace{ts}_{\in S} = \underbrace{tb}_{\in \mathfrak{a}} \in S \cap \mathfrak{a}$. $\square$

<u>Proposition:</u> Prime ideals in $S^{-1}R$ are of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p} \subsetneq R$ is a prime ideal with $\mathfrak{p} \cap S = \emptyset$.

<u>Proof</u> Let $\mathfrak{q} \subsetneq S^{-1}R$ be a prime ideal. By the proof of the previous Theorem, we know $\mathfrak{q} = S^{-1}\mathfrak{a}$ where $\mathfrak{a} = j_S^{-1}(\mathfrak{q})$.
Since $\mathfrak{q}$ is prime & j_S is a ring homomorphism, we know $\mathfrak{a}$ is a prime ideal of R. Since $\mathfrak{q} \subsetneq S^{-1}R$, we must have $\mathfrak{a} \cap S = \emptyset$
Conversely, given $\mathfrak{p} \subsetneq R$ prime with $\mathfrak{p} \cap S = \emptyset$, we want to show $S^{-1}\mathfrak{p} \subseteq S^{-1}R$ is a prime ideal.

- Properness follows since $\mathfrak{p} \cap S = \emptyset$
- $S^{-1}\mathfrak{p}$ is an ideal of $S^{-1}R$ by the Theorem.
- $\frac{a}{s} \cdot \frac{b}{t} \in S^{-1}\mathfrak{p}$ with $a, b \in R$ $s, t \in R$ we get

$\frac{ab}{1} = \frac{(st)}{1} \frac{ab}{st} \in S^{-1}\mathfrak{p} \Rightarrow ab \in \mathfrak{p} \xRightarrow{\mathfrak{p} \text{ prime}} a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

$\Rightarrow \frac{a}{s} \in S^{-1}\mathfrak{p}$ or $\frac{b}{t} \in S^{-1}\mathfrak{p}$.