Lecture 20: Localization & Noetherian rings

Recall: Rings & modules of fractions for R commutative ring

$S \subset R$ mult. closed set $\rightsquigarrow$ $S^{-1}R$ = another comm ring

M : R-module $\rightsquigarrow$ $S^{-1}M$: an $S^{-1}R$-module.

Prop: (1) Every ideal of $S^{-1}R$ is of the form $S^{-1}\mathfrak{a}$ for $\mathfrak{a} \subset R$ ideal & $S^{-1}\mathfrak{a} = S^{-1}R \iff S \cap \mathfrak{a} \neq \emptyset$.

(2) Prime ideals of $S^{-1}R \xleftrightarrow{1-to-1}$ prime ideals of A not meeting S

$(j_S(\mathfrak{p}))S^{-1}R = S^{-1}\mathfrak{p} \longleftarrow\!\shortmid \mathfrak{p}$

§1. Localization:

GOAL: Build suitable S where $S^{-1}R$ is a local ring (unique max l ideal)

Geometrically: study a space X in the neighborhood of a point.

$R = \{$polynomial functions $X \longrightarrow \mathbb{C}\}$.

Fix $\mathfrak{p} \subsetneq R$ a prime ideal and let $S = R \setminus \mathfrak{p}$.

Lemma: S is multiplicatively closed.

Pf/ • $1 \in S$ $(1 \notin \mathfrak{p})$ & • $0 \notin S$ $(0 \in \mathfrak{p})$

• $a, b \in S$ means $a, b \notin \mathfrak{p}$. So $ab \notin \mathfrak{p}$ because $\mathfrak{p}$ is prime $\Rightarrow ab \in S$. □

Def: $R_\mathfrak{p} := S^{-1}R$ is called the localization of R at the prime ideal $\mathfrak{p}$.

Proposition: $R_\mathfrak{p}$ is a local ring with unique maximal ideal $\mathfrak{p}R_\mathfrak{p}$.

Proof: Let $\mathfrak{b}$ be a proper ideal of $R_\mathfrak{p}$. By Prop (1) $\mathfrak{b} = S^{-1}\mathfrak{a}$ for $\mathfrak{a} \subset R$ ideal. If $\mathfrak{a} \cap S \neq \emptyset$, then $\mathfrak{b} = S^{-1}R$. Contr!

So $\mathfrak{a} \cap S = \emptyset$, meaning $\mathfrak{a} \subset \mathfrak{p}$. Hence $\mathfrak{b} \subseteq \mathfrak{p}(S^{-1}R)$

So every proper ideal of $R_\mathfrak{p}$ lies in $\mathfrak{p}R_\mathfrak{p}$. Thus $(R_\mathfrak{p}, \mathfrak{p}_\mathfrak{p})$ is local □

Obs: If R is a domain, $j_S: R \hookrightarrow \mathrm{Quot}(R) = (R\setminus\{0\})^{-1}R$

So $R \hookrightarrow R_{\mathfrak{p}} \hookrightarrow \mathrm{Quot}(R)$

Examples ① $R = \mathbb{Z}$ $\quad (p)$ is prime ideal $\rightsquigarrow S = \{b \in \mathbb{Z} : p \nmid b\}$

$\rightsquigarrow \mathbb{Z}_{(p)} = \{\frac{a}{b} : a, b \in \mathbb{Z} \ \gcd(a,b) = 1, \ p \nmid b\}$

② $R = \mathbb{C}[x]$ $\quad (x)$ is mxl ideal, so prime

$R_{(x)} = \{\frac{P}{Q} : P, Q \in \mathbb{C}[x] : x \nmid Q\}$

③ $R = \mathbb{C}[x,y]$ $\quad \mathfrak{m} = (x,y)$ is mxl ideal, so prime

$R_{(x,y)} = \{\frac{P}{Q} : P, Q \in \mathbb{C}[x,y] \ \ Q(0,0) \neq 0\}$

Def: Given M an R-module, we define its <u>localization at $\mathfrak{p}$</u> as $M_{\mathfrak{p}} = S^{-1}M$ where $S = R \setminus \mathfrak{p}$.

Q: What is $\ker(M \xrightarrow{i_S} S^{-1}M)$?

A: $\ker(i_S) = \{m \in M : \exists s \in S \text{ with } s \cdot m = 0 \text{ in } M\}$

Def: $\mathrm{Ann}(m) = \{r \in R : rm = 0\}$ (Annihilator of m)

Lemma: $\mathrm{Ann}(m)$ is an ideal of R. It is proper $\Leftrightarrow m \neq 0$.

Consequence: $m \in \ker(i_S) \Leftrightarrow \mathrm{Ann}(m) \cap S \neq \emptyset$.

Localizations are useful tools to decide when modules are trivial. More precisely:

Theorem: $M = \{0\} \Leftrightarrow M_{\mathfrak{p}} = 0 \quad \forall \mathfrak{p} \subsetneq R$ prime ideal

$\Leftrightarrow M_{\mathfrak{m}} = 0 \quad \forall \mathfrak{m} \subsetneq R$ mxl ideal

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear (mxl ideals are prime)

To finish, we prove (3) $\Rightarrow$ (1): We argue by contradiction. Pick $m \in M \setminus \{0\}$ & let $\mathfrak{a} = \mathrm{Ann}(m) \subsetneq R$. Pick $\mathfrak{m} \subset R$ maximal ideal with $\mathfrak{a} \subset \mathfrak{m}$. By hypotheses $M_{\mathfrak{m}} = 0$, so $\frac{m}{1} = 0$ in $M_{\mathfrak{m}}$ meaning $\exists\ s \in R \setminus \mathfrak{m}$ with $sm = 0$. This cannot happen since $(R \setminus \mathfrak{m}) \cap \mathrm{Ann}(m) = \emptyset$. $\square$

<u>Theorem</u>: Assume R is an integral domain. Then:

$$R = \bigcap_{\substack{\mathfrak{m} \text{ mxl} \\ \text{ideal}}} R_{\mathfrak{m}} = \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \text{ideal}}} R_{\mathfrak{p}}$$

Proof We know $R \hookrightarrow R_{\mathfrak{m}} \hookrightarrow \mathrm{Quot}(R)$

Write $\tilde{R} = \bigcap_{\mathfrak{m} \text{ mxl}} R_{\mathfrak{m}} \supseteq R$. We view $\tilde{R}/R$ as an R-module

Then: $\tilde{R} = R \iff \tilde{R}/R = 0$ as an R-module

Since $\tilde{R} \subset \mathrm{Quot}(R)$ we write $\bar{r} \in \tilde{R}/R$ as $\overline{\frac{a}{b}}$ with $\frac{a}{b} \in \mathrm{Quot}(R)$

- We want to show $a \in (b)$, so $\frac{a}{b} \in R$.

Consider $I = \{t \in R : t\frac{a}{b} \in R\} = \mathrm{Ann}(\overline{\frac{a}{b}})$

This means $\frac{ta}{b} = \frac{a'}{1}$ for $a' \in R$, ie $ta = a'b$

Thus $I = \{t \in R : ta \in R(b)\}$.

- If $I = (1)$, then $a = 1 \cdot a \in (b)$
- Otherwise, $\exists\ \mathfrak{m}$ mxl ideal of R with $I \subseteq \mathfrak{m} \subsetneq R$. Since $r - \frac{a}{b} \in R$, we have $\frac{a}{b} \in \tilde{R} \subseteq R_{\mathfrak{m}}$ So $\frac{a}{b} = \frac{a'}{b'}$ with $b' \notin \mathfrak{m}$

$b'a = a'b$ so $b' \in I \subseteq \mathfrak{m}$ <u>Contr!</u> $\square$

§2 Modules of fractions and their homomorphisms:

Fix R commutative ring & $S\subset R$ mult closed set.

Let M, N be two R-modules & set $f: M \longrightarrow N$ R-linear

Then $S^{-1}f: S^{-1}M \longrightarrow S^{-1}N$

$$\frac{m}{s} \longmapsto \frac{f(m)}{s}$$

Proposition: Let $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be a ses of R-linear maps between R-modules. Then, the following sequence of $S^{-1}R$-linear maps is exact:

$$0 \longrightarrow S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3 \longrightarrow 0.$$

Proof: (1) $\operatorname{Ker}(S^{-1}f) = \{\frac{m}{s} \in S^{-1}M_1 : \frac{f(m)}{s} = 0 \text{ in } S^{-1}M_2\}$

If $\frac{f(m)}{s} = 0$, then $f(m) = s \cdot \frac{f(m)}{s} = 0 \underset{m\in M}{\Rightarrow} m \in \operatorname{Ker} f = \{0\}$

Conclude: $\operatorname{Ker}(S^{-1}f) = \{0\}$.

(2) $S^{-1}g$ is surjective: Let $\frac{m_3}{s} \in S^{-1}M_3$, $m_3 \in M_3$, $s\in S$

Since g is surjective $\exists\, m_2 \in M_2$ st $g(m_2) = m_3$

So $\frac{m_3}{s} = \frac{g(m_2)}{s} = S^{-1}g\left(\frac{m_2}{s}\right)$. Conclude: $S^{-1}g$ is surjective.

(3) $\operatorname{Ker}(S^{-1}g) = \operatorname{Im}(S^{-1}f)$.

($\supseteq$) $(S^{-1}g)\circ(S^{-1}f)\left(\frac{m_1}{s}\right) = S^{-1}(g)\left(\frac{f(m_1)}{s}\right) = \frac{g(f(m_1))}{s} = \frac{0}{s} = 0$

So $\operatorname{Im}(S^{-1}f) \subseteq \operatorname{Ker}(S^{-1}g)$

($\subseteq$) Conversely, if $\frac{m_2}{s} \in \operatorname{Ker}(S^{-1}g)$ then $\frac{g(m_2)}{s} = 0$ so

$g(m_2) = s\,\frac{g(m_2)}{s} = 0$ so $m_2 \in \operatorname{Ker} g = \operatorname{Im} f$ so

$m_2 = f(m_1)$ for some $m_1 \in M_1$. Thus, $\frac{m_2}{s} = f\left(\frac{m_1}{s}\right) \in \operatorname{Im} S^{-1}f$. $\square$

Obs: Can use this to give an alternative proof of Thm on page 3.

Corollary: (1) Let $N \subset M$ be submodule over R.

Then $\dfrac{S^{-1}M}{S^{-1}N} \simeq S^{-1}(M/N)$. (as $S^{-1}R$-modules)

(2) In particular, for an ideal $\mathfrak{a} \subset R$, we have

$$S^{-1}R/S^{-1}\mathfrak{a} \simeq S^{-1}(R/\mathfrak{a}) \simeq \bar{S}^{-1}(R/\mathfrak{a}) \qquad \text{(as } S^{-1}R\text{-modules)}$$

where $\bar{S}$ = image of S under $R \longrightarrow R/\mathfrak{a}$.

Pf/ (1) Use $0 \to N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ ses of R-mod

Then $0 \to S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$ is ses of $S^{-1}R$-mod

(2) Need to show $S^{-1}(R/\mathfrak{a}) \simeq \bar{S}^{-1}(R/\mathfrak{a})$ $S^{-1}R$-mod iso

$$\frac{\bar{r}}{s} \longmapsto \frac{\bar{r}}{\bar{s}} \qquad \left(\frac{a}{b} \cdot \frac{\bar{r}}{s} = \frac{\overline{ar}}{bs}\right)$$

Well-def $\dfrac{\bar{r}}{s} = \dfrac{\bar{r}'}{s'} \iff \exists\, t \in S$ st $t(s'\bar{r} - s\bar{r}') = 0$ in $R/\mathfrak{a}$.

$\iff t(s'r - sr') \in \mathfrak{a}$ & $t \in S$ $\iff \bar{t}(\bar{s}'\bar{r} - \bar{s}\bar{r}') = 0$ in $R/\mathfrak{a}$ & $\bar{t} \in \bar{S}$. $\square$

§3. Finiteness properties of rings: Noetherian & Artinian rings

In 1888, Kronecker published his findings on "ideal = product of prime ideals" research. He a crucial assumption for ideals over polynomial rings: that they are finitely generated. This fact was only later proven by Hilbert (Hilbert Basis Thm). This property of rings ("every ideal is finitely generated") has the following axiomatization

Definition: A commutative ring R is called Noetherian if for every chain of ideals of R $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots$

there is $k \geq 0$ with $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \cdots$ [ACC = Ascending chain condition]

Theorem: The following conditions on a commutative ring R are equivalent:

(1) R is Noetherian

(2) Every nonempty set $\mathcal{S}$ of ideals of R has a maximal element

(3) Every ideal $\mathfrak{a} \subseteq R$ is finitely generated.

Proof: $\boxed{(1) \Rightarrow (2)}$ Let $\mathfrak{a}_0 \in \mathcal{S}$. If $\mathfrak{a}_0$ is not maximal, $\exists$ $\mathfrak{a}_1 \in \mathcal{S}$ with $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1$. Continuing in this fashion, we get an ascending chain of ideals $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots$ That doesn't stabilize. Contr! Then $\exists\, \mathfrak{a}_k$ maximal element of $\mathcal{S}$.

$\boxed{(2) \Rightarrow (3)}$ Let $\mathfrak{a}$ be an ideal. Consider the set

$$\mathcal{S} = \{ \mathfrak{a}' \subseteq \mathfrak{a} : \mathfrak{a}' \text{ is a fin. gen. ideal of } R \}$$

We order $\mathcal{S}$ by inclusion.

By (2), this set has a maximal element, say $\tilde{\mathfrak{a}}$.
If $\tilde{\mathfrak{a}} \subsetneq \mathfrak{a}$, pick $x \in \mathfrak{a} \setminus \tilde{\mathfrak{a}}$. Then $(\tilde{\mathfrak{a}}, x) \in \mathcal{S}$, contradicting the maximality of $\tilde{\mathfrak{a}} < (\tilde{\mathfrak{a}}, x)$. Hence $\tilde{\mathfrak{a}} = \mathfrak{a}$, which means $\mathfrak{a}$ is finitely generated since $\tilde{\mathfrak{a}} \in \mathcal{S}$.

$\boxed{(3) \Rightarrow (1)}$ Let $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots$ be a chain of ideals of R. Take $\mathfrak{a} = \bigcup_{i=0}^{\infty} \mathfrak{a}_i \subset R$.

By construction, $\mathfrak{a}$ is an ideal of R, thus finitely generated by elements $a_1, \ldots, a_n \in \mathfrak{a}$. Now, each $a_\ell \in \mathfrak{a}_{j_\ell}$ for some $j_\ell \geq 0$.
Thus, $\mathfrak{a} = \mathfrak{a}_j$ for $j = \max\{j_1, j_2, \ldots, j_n\}$ and so $\mathfrak{a} = \mathfrak{a}_j = \mathfrak{a}_{j+1} = \cdots$ The chain terminates, so A is Noetherian.

Corollary: (1) Principal ideal domains are Noetherian (eg $\mathbb{Z}$, $\mathbb{C}[x]$)

(2) If $f: A \longrightarrow B$ is a ring homomorphism with A, B commutative. Assume f is surjective. If A is Noetherian, so is B.

(3) Rings of fractions of Noetherian rings are Noetherian. In particular, localizations preserve Noetherianness.

Proof (2): If $\mathfrak{b} \subset B$ is an ideal, $\mathfrak{a} = f^{-1}(\mathfrak{b}) \subset A$ is an ideal, so $\mathfrak{a} = (a_1, \ldots, a_n)$. Then $\mathfrak{b} \underset{f \text{ surj}}{=} f(\mathfrak{a}) = (f(a_1), \ldots, f(a_n))$.

⚠ Subrings of Noetherian rings need not be Noetherian.

Ex: Take $R = \mathbb{C}[x_1, x_2, \ldots] = \bigcup_{n \in \mathbb{N}} \mathbb{C}[x_1, x_2, \ldots, x_n]$

R is not Noetherian since

$$\mathfrak{a}_1 = (x_1) \subsetneq \mathfrak{a}_2 = (x_1, x_2) \subsetneq \cdots \quad \text{never terminates.}$$

But R is a domain & $R \hookrightarrow \mathrm{Quot}(R) = \text{field}$.

Since the only ideals of $\mathrm{Quot}(R)$ are (0) & (1), we get that $\mathrm{Quot}(R)$ is Noetherian.

Hilbert Basis Thm: If A is Noetherian, so is $A[x]$.

Hence $\mathbb{Z}[x_1, \ldots, x_n]$, $\mathbb{K}[x_1, \ldots, x_n]$ are Noetherian for any field $\mathbb{K}$.