Lecture 21: Noetherian modules & Hilbert Basis Thm

§1 Noetherian rings:

Definition: A commutative ring R is called **Noetherian** if for every chain of ideals of R $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots\cdots$
there is $k \geq 0$ with $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \cdots$ [ACC = Asc chain condition]

Theorem 1: Fix R a commutative ring. TFAE:

(1) R is Noetherian

(2) Every nonempty set $\mathcal{S}$ of ideals of R has a maximal element

(3) Every ideal $\mathfrak{a} \subseteq R$ is finitely generated.

Corollary 1: (1) Principal ideal domains are Noetherian (eg $\mathbb{Z}, \mathbb{C}[x]$)

(2) If $f: A \longrightarrow B$ is a ring homomorphism with A, B commutative. Assume f is surjective. If A is Noetherian, so is B.

(3) Rings of fractions of Noetherian rings are Noetherian. In particular, localizations preserve Noetherianness.

Proof (2): If $\mathfrak{b} \subset B$ is an ideal, $\mathfrak{a} = f^{-1}(\mathfrak{b}) \subset A$ is an ideal, So $\mathfrak{a} = (a_1, \ldots, a_n)$. Then $\mathfrak{b} \underset{f \text{ surj}}{=} f(\mathfrak{a}) = (f(a_1), \ldots, f(a_n))$.

⚠ Subrings of Noetherian rings need not be Noetherian.

Ex: Take $R = \mathbb{C}[x_1, x_2, \ldots] = \bigcup_{n \in \mathbb{N}} \mathbb{C}[x_1, x_2, \ldots, x_n]$

R is not Noetherian since

$\mathfrak{a}_1 = (x_1) \subsetneq \mathfrak{a}_2 = (x_1, x_2) \subsetneq \cdots\cdots$ never terminates.

But R is a domain & $R \hookrightarrow \mathrm{Quot}(R)$ = field.

Since the only ideals of $\mathrm{Quot}(R)$ are (0) & (1), we get that $\mathrm{Quot}(R)$ is Noetherian.

Hilbert Basis Thm: If R is Noetherian, so is $R[x]$.

Hence $\mathbb{Z}[x_1,\dots,x_n]$, $\mathbb{K}[x_1,\dots,x_n]$ are Noetherian for any field $\mathbb{K}$.

. To prove this result, we'll need the notion of Noetherian modules.

§2. Noetherian modules:

Fix R = commutative ring. & M an R-module.

Def: We say M is <u>Noetherian</u> if it satisfies the ascending chain condition for submodules:

"Every chain of submodules $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ stabilizes, ie $\exists\, l \geq 0$ st $M_l = M_{l+1} = \dots$"

We have the following analog of Theorem 1:

Theorem 2: Fix R a commutative ring & M an R-module. TFAE:

(1) M is Noetherian

(2) Every nonempty set $\mathcal{S}$ of submodules of M has a maximal element

(3) Every submodule of M is finitely generated.

The proof is exactly the same as that of Thm 1.

Corollary 2: Let $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be a ses of R-modules. Then: M_2 is Noetherian if, and only if, M_1 & M_3 are.

Proof: $\Rightarrow$) submodules of M_1 are submodules of M_2 via f.

——— $N_3 \leq M_3$ come from $g^{-1}(N_3) \subseteq M_2$ submodules

$g^{-1}(N_3) = \langle m_1, \dots m_l \rangle$, then $N_3 = \langle g(m_1), \dots, g(m_l) \rangle$ by surjectivity of g.

$\Leftarrow$) Pick N a submodule of M_2 then $g(N) \subseteq M_3$ is a subm.

So $g(N) = \langle m_1, \dots, m_s \rangle$. Pick $n_1, \dots n_s \in N$ with $g(n_1) = m_1, \dots, g(n_s) = m_s$.

Next Take $f^{-1}(N \cap f(M_1)) = N_1 \subseteq M_1$ submodule, so its finitely generated $N_1 = \langle q_1, \dots q_e \rangle$

Then $n'_1 = f(q_1), \dots, n'_e = f(q_e)$. & $N \cap f(M_1) = \langle n'_1, \dots, n'_e \rangle$

<u>Claim</u>: $N = \langle n_1, \dots, n_s, n'_1, \dots, n'_e \rangle$

Pf/ Pick $n \in N$, so $g(n) \in g(N) = \langle m_1, \dots, m_s \rangle$

That is $g(n) = a_1 m_1 + \dots + a_s m_s = a_1 g(n_1) + \dots + a_s g(n_s)$

$= g(a_1 n_1 + \dots + a_s n_s)$ $\quad a_1, \dots a_s \in R$

This means $n - a_1 n_1 - \dots - a_s n_s \in \operatorname{Ker} g = \operatorname{Im} f$, so

$n - a_1 n_1 - \dots - a_s n_s \in N \cap f(M_1) = \langle n'_1, \dots, n'_e \rangle$

Conclude $n \in \langle n_1, \dots n_s, n'_1, \dots n'_e \rangle$ $\square$

<u>Note</u>: (1) R is an R-module. Then: R is a Noetherian R-module if, and only if R is a Noetherian ring.

(2) Recall: Subrings of a Noetherian ring need not be Noetherian

(3) A finite direct sum of Noetherian modules is Noetherian

(<u>Hint</u>: Induct on the number of summands & use Corollary 2)

(4) M: Noetherian R-module $\Rightarrow S^{-1}M$ is a Noetherian $S^{-1}R$-mod.
S mult closed set of R

<u>Proposition</u>: Let R be a Noetherian ring & M an R-module. Then M is Noetherian if & only if M is finitely generated, ie $\exists\, x_1, \dots, x_e \in M$ st every x in M can be written (no necessarily uniquely) as $x = a_1 x_1 + \dots + a_e x_e$ for $a_1, \dots, a_e \in R$.

Proof ($\Rightarrow$) View M as a submodule of M & use Theorem 2.

($\Leftarrow$) As M is finitely generated, eg $M = \langle x_1, \dots, x_\ell \rangle$ we have $R \xrightarrow{f_i} M$ morphism of R-modules. By the
$a_i \longmapsto a_i x_i$

universal property of $\underbrace{R \oplus \dots \oplus R}_{\ell \text{ copies}}$ we have a unique

$f: \bigoplus_{i=1}^{\ell} R \longrightarrow M$ R-linear map st

$(a_1, \dots, a_\ell) \longmapsto \sum_{i=1}^{\ell} a_i x_i$

$$\begin{tikzcd} \bigoplus_{i=1}^{\ell} R \arrow[dr, "f"] & \\ R \arrow[u, "j_i"] \arrow[r, "f_i"'] & M \end{tikzcd} \quad \forall i$$

Furthermore, we get a ses of R-modules:

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \bigoplus_{i=1}^{\ell} R \longrightarrow M \longrightarrow 0$$

But R is Noetherian, so $\bigoplus_{i=1}^{\ell} R$ is also a Noetherian R-module. Again by Corollary 2: $\operatorname{Im} f = M$ is Noetherian $\square$.

Examples ① $R = \mathbb{K}$ a field, $M = \mathbb{K}$-vector space

M Noetherian $\Longleftrightarrow$ $\dim_{\mathbb{K}} M < \infty$

② $R = \mathbb{K}[x]$ (Noetherian ring, since it's a PID)

But $M = \mathbb{K}[x,y]$ is not a Noetherian R-module.

It will be a Noetherian ring! $(1, y, y^2, \dots)$ not fg R-submod.

③ An example of non-Noetherian ring:

$R = \{$ continuous $\mathbb{C}$-valued functions on $\mathbb{R}$ $\}$

$F_n = [-\frac{1}{n}, \frac{1}{n}]$ $n \geq 1$ (nested chain of intervals with $|F_n| \searrow 0$)

$\mathfrak{a}_n = \{ f \in R \mid f|_{F_n} \equiv 0 \}$ is an ideal in R.

$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \mathfrak{a}_3 \subsetneq \cdots\cdots$ is a strictly increasing chain of ideals.

So R is Not noetherian..

§3 Hilbert Basis Theorem:

Theorem 3: If R is commutative and Noetherian, so is $R[X]$.

Proof: We will show that every ideal of $R[X]$ is finitely generated.
Let $\mathfrak{b} \subset R[X]$ be an ideal. For every $f(x) \in R[X]$ we let $L.T(f) \in R$ be the leading coefficient of f

$$\underset{\neq 0}{f} = a_0 + a_1 X + \cdots + a_n x^n \rightsquigarrow LT(f) := a_n \in R$$

with $a_n \neq 0$

We define $LT(0) = 0$.

Claim: $\mathfrak{a} = \{LT(f) : f \in \mathfrak{b}\} \subset R$ is an ideal.

Pf/ (1) $0 \in \mathfrak{a}$ since $LT(0) = 0$ & $0 \in \mathfrak{b}$

(2) $a\,LT(f) \in \mathfrak{a}$ $\forall a \in R$ & $f \in \mathfrak{b}$. Clear if $aLT(f) = 0$

Otherwise $aLT(f) = LT(af) \in \mathfrak{a}$ $\quad \left(a\sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} (aa_i) x^i\right)$

$a_n \neq 0$ & $aa_n \neq 0$.

So $-LT(f) \in \mathfrak{a}$ if $f \in \mathfrak{b}$.

(3) $LT(f) + LT(g) \in \mathfrak{a}$ if $LT(f)$ & $LT(g)$ do

. If $LT(f) = -LT(g)$ we know $0 \in \mathfrak{a}$, so assume $LT(f) + LT(g) \neq 0$.

if $\underset{k}{\underset{\parallel}{\deg(f)}} \leq \underset{l}{\underset{\parallel}{\deg g}}$ then $x^{l-k} f \in \mathfrak{b}$ & $g \in \mathfrak{b}$

so $LT(f) + LT(g) \underset{\substack{\uparrow \\ \text{no cancellation occurs}}}{=} LT(\underbrace{x^{l-k} f + g}_{\in \mathfrak{b} \text{ (since } f \& g \in \mathfrak{b})}) \in \mathfrak{a}$

Since R is Noetherian, we know $\mathfrak{a}$ is f.g. by $a_1, \dots a_\ell$ with $a_i \neq 0 \; \forall i$. For each $j = 1, \dots, \ell$ pick $f_j \in \mathfrak{b}$ with $\boxed{a_j = LT(f_j).}$

Let $r = \max_{1 \leq j \leq \ell} \deg(f_j) \geq 0$. Let $M \subset R[x]$ be the R-submodule generated by $\{1, x, \dots, x^{r-1}\}$ (so M is the set of polynomials of degree $< r$) Since R is Noetherian & M is f.g, then M is noetherian.

Now $\mathfrak{b} \cap M \subset M$ is a submodule of M so it's also finitely generated say by $\{b_1, \dots, b_k\}$.

<u>Claim 2</u>: $\mathfrak{b} = \langle b_1, \dots, b_k, f_1, \dots f_\ell \rangle$

Pf/ Pick $f \in \mathfrak{b}$. If $\deg(f) < r$, then $f \in \mathfrak{b} \cap M$ & hence $f \in \langle b_1, \dots, b_k \rangle$. Otherwise, we proceed by induction on $\deg(f) \geq r$.

Let $a = LT(f)$ with $\deg(f) = d \geq \deg(f_j) =: d_j \; \forall j$

Since $a \in \mathfrak{a}$, we have $a = r_1 a_1 + \dots + r_\ell a_\ell$ for suitable $r_1, \dots r_\ell$

Thus, $g = f - \sum r_j x^{d-d_j} f_j \in \mathfrak{b}$ & $\deg g < \deg f$.

- If $\deg f = r$, then $g \in \mathfrak{b} \cap M$ and we are done: Indeed:

$$g = f - \sum r_j x^{d-d_j} f_j = c_1 b_1 + \dots + c_k b_k$$

So $f \in \langle b_1, \dots b_k, f_1, \dots, f_\ell \rangle$

- If $\deg f > r$, then $\deg g < \deg f$ & $g \in \mathfrak{b}$. By IH, $g \in \langle b_1, \dots, b_k, f_1, \dots, f_\ell \rangle$, so the same holds for f. $\square$