# Lecture 22: Artinian Rings

<u>Last Time</u>: discussed Noetherian modules & Hilbert Basis Theorem
(ACC)
<u>TODAY</u>: Artinian rings, defined using descending chains.

**Q** Why study Artinian rings?

**A**: Geometrically Artinian rings correspond to finite collections of fat points (ie, points with multiplicities)

## §1 Definition & first properties:

<u>Definition</u>: Let $R$ be a commutative ring. We say that $R$ is <u>Artinian</u> (after Emil Artin) if every descending chain of ideals $\mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \dots$ stabilizes, ie $\exists\, l \geq 0$ with $\mathfrak{a}_l = \mathfrak{a}_{l+1} = \cdots$ (Descending Chain Condition)

Ex: ① $R = \mathbb{K}$ field is Artinian

② $R = \mathbb{K}[x]/(x^n)$ —— ( Ideals are $\mathbb{K}$-subspaces & $\dim_{\mathbb{K}} R = n$)

<u>Lemma 1</u>: Let $\mathcal{S}$ be non-empty set of ideals in an Artinian ring. Then $\mathcal{S}$ has minimal elements (with respect to inclusion)

<u>Proof</u>: (Same idea as with Noetherian rings)

Let $\mathfrak{a}_0 \in \mathcal{S}$. If $\mathfrak{a}_0$ is minimal, we are done. Otherwise, we find $\mathfrak{a}_1 \in \mathcal{S}$ with $\mathfrak{a}_0 \supsetneq \mathfrak{a}_1$. As $R$ is Artinian, this process must stop and we will arrive at a minimal element of $\mathcal{S}$. □

<u>Lemma 2</u>: Artinian property is preserved under quotients by ideals

Pf/ Let $\mathfrak{a} \subseteq R$ be an ideal and $R$ be Artinian

Then $\tilde{R} = R/\mathfrak{a}$ is also Artinian since ideals in $\tilde{R}$ correspond to ideals in $R$ containing $\mathfrak{a}$.

So the DCC for $R$ yields the DCC for $\tilde{R}$. □

**Proposition 1**: Let $R$ be an Artinian commutative ring. Then:

(i) Every prime ideal in $R$ is maximal.

(ii) There are only finitely many maximal ideals in $R$.

**Proof**: (i) Let $\mathfrak{p} \subsetneq R$ be a prime ideal. Then $R/\mathfrak{p}$ is an Artinian integral domain. (by Lemma 2)

Now that $x \in R/\mathfrak{p} \setminus \{0\}$ & consider the descending chain of ideals in $R/\mathfrak{p}$:

$$(x) \supseteq (x^2) \supseteq (x^3)$$

Since it eventually stabilizes, $\exists\, k \geq 1$ with $(x^k) = (x^{k+1})$

ie $x^k = y x^{k+1}$ for $y \in R/\mathfrak{p}$.

$\Rightarrow x^k(1 - xy) = 0$

As $R/\mathfrak{p}$ is a domain and $x \neq 0$ we have $1 = xy$, so

$x$ is a unit.

We conclude $(R/\mathfrak{p})^\times = R/\mathfrak{p} \setminus \{0\}$, so $R/\mathfrak{p}$ is a field

This means $\mathfrak{p}$ is a maximal idea of $R$. $\square$

(ii) Let $\boxed{\mathcal{J} = \text{set of ideals that are finite intersections of maximal ideals of } R}$

- $\mathcal{J} \neq \emptyset$ since maximal ideals $\mathfrak{m}$ exist & $\mathfrak{m} \in \mathcal{J}$.
- By the Artinian condition & Lemma 1, $\mathcal{J}$ has a minimal element $\mathfrak{a} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell$

<u>Claim</u>: $\{\text{Maximal ideals in } R\} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$

Pf/ Pick $\mathfrak{m} \subsetneq R$ maximal ideal, then $\mathfrak{m} \cap \mathfrak{a} \in \mathcal{J}$

& $\mathfrak{m} \cap \mathfrak{a} \subseteq \mathfrak{a}$. By the minimality of $\mathfrak{a}$, we have $\mathfrak{a} = \mathfrak{m} \cap \mathfrak{a}$ so

$$\mathfrak{m} \cap \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell \underset{\text{prime}}{\subseteq} \mathfrak{m}$$

By Prime Avoidance (Lecture 17, Thm 2, page 7) $\exists j$ st $\mathfrak{m}_j \subseteq \mathfrak{m}$

Since $\mathfrak{m}_j$ & $\mathfrak{m}$ are both maximal, we have $\mathfrak{m}_j = \mathfrak{m}$. □

## §2. Nilradicals of Artinian rings

**Corollary 1:** Let $\mathcal{N}$ be the ideal of nilpotent elements (ie the nilradical of $R$). If $R$ is Artinian, then

$$\mathcal{N} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell$$

where $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$ is the list of maximal ideals of $R$.

Pf/ $\mathcal{N} \overset{(*)}{=} \bigcap_{\substack{\mathfrak{p} \subseteq R \\ \text{prime}}} \mathfrak{p} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell$ because all prime ideals are maxl in the Artinian case.

(*) by Problem 4 HW7. □

Obs Problem 9 in HW7 says $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell =: \mathcal{J} =$ Jacobson radical

$$\mathcal{J} = \{x \in R : 1 - xy \text{ is a unit } \forall y \in R\}$$

- From now on, we assume $R$ is commutative & Artinian.

**Proposition 2:** The ideal $\mathcal{N} \subset R$ (Artinian) is nilpotent, ie $\exists\, n \geq 0$ such that $\mathcal{N}^n = (0)$.

**Proof:** We consider the chain of ideals of $R$

$$\mathcal{N} \supseteq \mathcal{N}^2 \supseteq \mathcal{N}^3 \supseteq \cdots$$

Since $R$ is Artinian, $\exists\, n \in \mathbb{N}$ with $\boxed{\mathfrak{a}} := \mathcal{N}^n = \mathcal{N}^{n+1} \ldots$

If $\mathfrak{a} = (0)$ we are done. So assume $\mathfrak{a} \neq (0)$. Then, we consider $\boxed{\mathcal{S} = \text{set of all ideals } \mathfrak{b} \subseteq R \text{ st } \mathfrak{a}\mathfrak{b} \neq (0)}$

- $\mathcal{S} \neq \emptyset$ since $\mathfrak{a}^2 = \mathfrak{a} \neq (0)$ so $\mathfrak{a} \in \mathcal{S}$.
- Pick $I \in \mathcal{S}$ minimal element. As $I \cdot \mathfrak{a} \neq (0)$, there is $x \in I$ st $x\mathfrak{a} \neq (0)$ so $(x) \in \mathcal{S}$. But $(x) \subseteq I$

By minimality, $(x) = I$.

But $(x\mathfrak{a})\mathfrak{a} = x\mathfrak{a}^2 = x\mathfrak{a} \neq \emptyset$ so $x\mathfrak{a} \in \mathcal{S}$

Once again $x\mathfrak{a} \subseteq (x)$, so minimality gives $(x) = x\mathfrak{a}$

This means $\exists\, y \in \mathfrak{a}$ st $x = xy$, ie

$x = xy = xy^2 = \cdots\cdots$

Since $y \in \mathfrak{a} \subset \mathcal{N}$ we have that $y$ is nilpotent.

ie $\exists\, m \in \mathbb{N}$ with $y^m = 0$.

<u>Conclusion</u>: $x = xy^m = 0$, contradicting $x\mathfrak{a} = I\mathfrak{a} \neq (0)$

$\square$