Lecture 24: Artinian Rings & Primary Decomposition

Last Time: $\boxed{\text{Artinian} \Longrightarrow \text{dimension } 0 \ \& \ \text{Noetherian}}$ (*)

Structure Theorem: finitely many max'l ideals $(m_1, \ldots, m_\ell)$ and
$R \simeq R/m_1^n \times \cdots \times R/m_\ell^n$ (each R/m_j^n is Artinian & local)

TODAY we will show the converse to (*). The proof is based on "primary decomposition".

§1. Definition & examples:

Fix R to be any commutative ring

Definition: An ideal $q \subsetneq R$ is primary if for any $a, b \in R$ we have " $ab \in q$ & $b \notin q \Longrightarrow a^n \in q$ for some $n \geq 1$."

Obs: Equivalently, every zero divisor in R/q is nilpotent. (HW9)

Recall: The radical of an ideal $\mathfrak{a}$ in R is
$$r(\mathfrak{a}) = \sqrt{\mathfrak{a}} = \{ a \in R \mid a^n \in \mathfrak{a} \text{ for some } n \text{ in } \mathbb{N} \}$$

Lemma: $q \subseteq R$ primary $\Longrightarrow r(q)$ is prime

Pf/ $a, b \in \mathfrak{p} = r(q) \Rightarrow a^k b^k \in q$ for some $k > 0$

$\underset{q \text{ primary}}{\Longrightarrow}$ either $\underbrace{b^k \in q}_{\Downarrow \; b \in \mathfrak{p}}$ or $\underbrace{(a^k)^n \in q \text{ for some } n \geq 1.}_{\Downarrow \; a \in \mathfrak{p}}$ □

Obs: The difference between q & $\mathfrak{p} = r(q)$ is algebraic and highlights the difference between a fat point (point with multiplicity) vs the point as a set. (The "algebraic part" of Alg Geometry)

Examples: ① $R = \mathbb{K}[x]$ (x^n) is primary. $r((x^n)) = (x)$ prime
$ab \in (x^n)$ then $x | a$ or $x | b$. This forces
$a^n \in (x^n)$ or $b^n \in (x^n)$. (if $x^n \nmid b$, then $x | a$).

② $R = \mathbb{K}[x,y]$ $\quad \mathfrak{q} = (x, y^2)$ is primary$^{(*)}$ but NOT a power of a prime ideal.

$$\mathfrak{p} = r(\mathfrak{q}) = (x,y) \quad \text{and} \quad \mathfrak{p}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p} \quad (x \notin \mathfrak{p}^2)$$

(*) $fg \in \mathfrak{q}$ write
$$f = a_0 + x f_{1(x,y)} + y f_{2(y)} \qquad a_0 \in \mathbb{K}$$
$$g = b_0 + x g_{1(x,y)} + y g_{2(y)} \qquad b_0 \in \mathbb{K}$$

$g \notin \mathfrak{q}$ means $b_0 \neq 0$ or $(f_3(0) \neq 0$ and $b_0 = 0)$

• Case 1: $b_0 \neq 0$

$$\Rightarrow fg = a_0 b_0 + x(g f_1 + a_0 g_1 + g_1 y f_{2(y)}) + y(f_2 g + a_0 g_{2(y)})$$

Since $fg \in (x, y^2)$, then $a_0 b_0 = 0$, so $a_0 = 0$.

$$\Rightarrow \underbrace{fg}_{\in \mathfrak{q}} = \underbrace{x(f_1 g + g_1 f_{2(y)} g_1) + y^2 f_{2(y)} g_{2(y)}}_{\in \mathfrak{q}} + b_0 y f_{2(y)}$$

$\Rightarrow$ $b_0 y f_{2(y)} \in \mathfrak{q}$ so $b_0 y f_{2(y)} = x h_{1(x,y)} + y^2 h_{2(x,y)}$

Evaluate at $x=0$ to get $b_0 y f_{2(y)} = y^2 h_2(0,y)$

$\Rightarrow$ $y \mid f_2(y)$ so $f = 0 + x f_{1(x,y)} + y^2 \underbrace{\frac{f_2}{y}}_{\in R} \in \mathfrak{q}$.

• Case 2: $b_0 = 0$ & $y \nmid g_2(y)$

$$\Rightarrow \underbrace{fg}_{\in \mathfrak{q}} = \underbrace{x(a_0 g_1 + x f_1 g_1 + y f_{2(y)} g_1 + y f_1 g_{2(y)}) + y^2 f g_2}_{\in \mathfrak{q}} + y a_0 g_{2}(y)$$

$\Rightarrow$ $y a_0 g_2(y) \in \mathfrak{q}$ so $y a_0 g_{2(y)} = x h_{1(x,y)} + y^2 h_{2(x,y)}$

$\xRightarrow[x=0]{}$ $y a_0 g_{2(y)} = y^2 h_2(0,y)$ So $\underbrace{y \mid g_2(y)}_{\text{Contra!}}$ or $\boxed{a_0 = 0}$

$\Rightarrow f = x f_1 + y f_{2(y)} \Rightarrow f^2 = x^2 f_1^2 + y^2 f_{2(y)}^2 + 2xy f_1 f_2$
$\in (x, y^2) = \mathfrak{q}$.

Note: $r(\mathfrak{q}) = (x,y)$ is maximal in $\mathbb{K}[x,y]$. $\square$

This last example is more general (see HW9)

<u>Proposition</u>: If R is commutative & $r(\mathfrak{q})$ is maximal then $\mathfrak{q}$ is primary.

<u>Example</u> 3 $R = \mathbb{K}[x,y,z]/(xy-z^2) \supset \mathfrak{p} = (\bar{x}, \bar{z})$

$\mathfrak{p}$ is a prime ideal but $\mathfrak{p}^2$ is not primary.

- $R/\mathfrak{p} = \mathbb{K}[x,y,z]/(x,z,xy-z^2) = \dfrac{\mathbb{K}[x,y,z]}{(x,z)} = \mathbb{K}[y]$ integral domain
- $\mathfrak{p}^2 = (\bar{x}^2, \bar{z}^2, \bar{x}\bar{z})$

$\bar{z}^2 = \bar{y}\bar{x} \in \mathfrak{p}^2$ & $\bar{x} \notin \mathfrak{p}^2$ but $\bar{y} \notin r(\mathfrak{p}^2)$. (Exercise)

⚠ We do have $\bar{y} \notin \mathfrak{p}^2$ but $\bar{x}^2 \in \mathfrak{p}^2$, i.e., the definition of primary is not symmetric in f & g.

<u>Summary of examples</u>:

- $\mathfrak{q}$ primary $\not\Rightarrow$ $\mathfrak{q}$ = power of a prime ideal
- $\mathfrak{p}$ prime $\not\Rightarrow$ $\mathfrak{p}^n$ primary.
- $r(\mathfrak{q})$ is maximal $\Rightarrow$ $\mathfrak{q}$ primary

§ 2. Irreducible ideals:

<u>Def</u>: An ideal $\mathfrak{a} \subseteq R$ is <u>irreducible</u> if $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{b}, \mathfrak{c} \subseteq R$ ideals, then $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$.

Terminology comes from <u>topology</u>: if $R = \mathbb{C}[x_1, \ldots x_n]$, then $\mathfrak{a}$, $\mathfrak{b}$ & $\mathfrak{c}$ define closed sets in $\mathbb{C}^n$ (solutions to polynomials

in each ideal: $V(\mathfrak{a})$, $V(\mathfrak{b})$ & $V(\mathfrak{c})$. Moreover:
$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ translates to $V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{c})$
So we can decompose $V(\mathfrak{a})$.

<u>Lemma</u>: Assume R is <u>Noetherian</u>. Then:

(i) Every ideal in R is a finite intersection of irreducible ideals.

(ii) Irreducible $\Rightarrow$ Primary

<u>Proof</u>: (i) Consider $\Sigma = \{ \mathfrak{a} \subseteq R$ ideal : $\mathfrak{a}$ is <u>not</u> a finite intersection of irred. ideals$\}$

. If $\Sigma \neq \emptyset$ it must have a maximal element (R Noeth)
say $\mathfrak{a} \in \Sigma$ is this mxl element.

Since $\mathfrak{a}$ is not irreducible (otherwise $\mathfrak{a} \notin \Sigma$), then
$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{a} \subsetneq \mathfrak{b}$ & $\mathfrak{a} \subsetneq \mathfrak{c}$ ideals.

Now $\mathfrak{b}, \mathfrak{c} \notin \Sigma$ by maximality of $\mathfrak{a}$, so

$$\begin{aligned} \mathfrak{b} &= \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_k \\ \mathfrak{c} &= \mathfrak{c}_1 \cap \cdots \cap \mathfrak{c}_\ell \end{aligned} \quad \text{with } \mathfrak{b}_i, \mathfrak{c}_j \text{ irreducible}$$

$\Rightarrow \mathfrak{a} = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_k \cap \mathfrak{c}_1 \cdots \cap \mathfrak{c}_\ell \notin \Sigma$ Contr!

<u>Conclude</u>: $\Sigma = \emptyset$.

(ii) Fix $\mathfrak{a} \subsetneq R$ irreducible ideal.

Working with $\tilde{R} = R/\mathfrak{a}$, we may assume (0) is an irreducible ideal
still Noetherian

Let $xy \in (0)$, $y \notin (0)$ ie $xy = 0$ & $y \neq 0$. We want to prove $x^n = 0$ for some $n > 0$.

Consider the chain of ideals:

$$\operatorname{Ann}(x) \subseteq \operatorname{Ann}(x^2) \subseteq \cdots \subseteq \operatorname{Ann}(x^i) \subseteq \cdots$$

$[\operatorname{Ann}(z) = \{ r \in R \mid rz = 0\} \underset{\text{ideal}}{\subseteq} R]$.

Since R is Noetherian, $\exists\, n>0$ st $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1}) = \cdots$

Claim: $(0) = (x^n) \cap (y)$.

Pf/ $\supseteq$) $a \in (y) \Rightarrow ax = 0$ (since $xy=0$)
$a \in (x^n) \Rightarrow a = bx^n$ $\Big\} \Rightarrow bx^{n+1} = 0$.

But $bx^{n+1} = 0 \Rightarrow b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$ so $bx^n = 0$

So $a = bx^n = 0$.

Since (0) is irreducible and $(y) \neq (0)$, we conclude $(x^n) = (0)$, ie $x^n = 0$ as required □

This lemma is referred to as "Primary Decomposition for Noetherian rings." We will come back to this next time.

§3. Characterization of primary ideals:

Fix R Noetherian & commutative. Let $\mathfrak{a} \subsetneq R$ be an ideal.

Write $\mathfrak{a} = q_1 \cap \cdots \cap q_\ell$ (a primary decomposition of $\mathfrak{a}$)

irreducible ($\Rightarrow$ primary)

Let $\mathfrak{p}_i = r(q_i)$ be the corresponding prime ideals.

Lemma: If $\mathfrak{p} \subsetneq R$ is a prime ideal, minimal among the set of prime ideals containing $\mathfrak{a}$, then $\mathfrak{p} = \mathfrak{p}_i$ for some $i=1,\dots,\ell$

Pf/ By Theorem 2 of Prime Avoidance (Lecture 17), we have $q_1 \cap \cdots \cap q_\ell \subseteq \underset{\text{prime}}{\mathfrak{p}} \Rightarrow q_i \subseteq \mathfrak{p}$ for some i.

Hence $\mathfrak{p}_i = r(q_i) \subseteq r(\mathfrak{p}) = \mathfrak{p}$, but

$$\mathfrak{a} \subseteq \underset{\text{prime}}{\mathfrak{p}_i} \subseteq \mathfrak{p} \quad \& \ \mathfrak{p} \text{ minimal} \Rightarrow \mathfrak{p}_i = \mathfrak{p}.$$

□

Def. The minimal primes of R are the prime ideals of R, minimal with respect to inclusion

Corollary: There are only finitely many minimal primes over any given ideal $\mathfrak{a}$ of a Noetherian ring R. ($\longleftrightarrow$ min primes in $R/\mathfrak{a}$)