Lecture 25: Primary Decomposition II

R commutative

Recall
- $\mathfrak{q}$ primary ideal if $xy \in \mathfrak{q}$ & $y \notin \mathfrak{q} \Rightarrow x^n \in \mathfrak{q}$ for some n
- $\mathfrak{q}$ irred ideal if $\mathfrak{q} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{b}, \mathfrak{c}$ ideals $\Rightarrow \mathfrak{q} = \mathfrak{b}$ or $\mathfrak{q} = \mathfrak{c}$.

Lemma: $\mathfrak{q}$ primary $\Rightarrow r(\mathfrak{q})$ prime

Obs:
1. $\mathfrak{q}$ primary $\not\Rightarrow$ $\mathfrak{q}$ is a power of a prime
2. $\mathfrak{p}$ prime $\not\Rightarrow$ $\mathfrak{p}^n$ is primary
3. $r(\mathfrak{q})$ maximal $\Rightarrow$ $\mathfrak{q}$ is primary

Prop: R Noetherian : (i) Every ideal is a finite intersection of irred. ideals
(ii) Irred $\Rightarrow$ Primary.

Consequence: "Primary Decomposition" for Noetherian rings.

- $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ & $\mathfrak{p}$ minimal among prime ideals containing $\mathfrak{a}$ then $\mathfrak{p} = r(\mathfrak{q}_i)$ for some i

Consequence : $\mathfrak{a}$ has finitely many minimal associate primes.
$(= \mathrm{Min}(\mathfrak{a}))$

§1. Noetherian + dim 0 $\Rightarrow$ Artinian

Theorem 1: Let R be a commutative Noetherian ring R of dim 0 (that is, every prime ideal is maximal). Then, R is Artinian.

Proof : By the Corollary, R has finitely many minimal primes. By our dim 0 assumption, these are maximal ideals. Call them $\mathfrak{m}_1, \ldots, \mathfrak{m}_e$.

- Any other maximal ideal $\mathfrak{m}$ must contain a minimal prime, namely $\mathfrak{m}$ itself. Indeed, if $\mathfrak{m}$ is not minimal $\exists\, \mathfrak{p} \subsetneq R$ prime ($\Rightarrow$ mxl) with $\mathfrak{p} \subseteq \mathfrak{m}$. But $\mathfrak{p}$ & $\mathfrak{m}$ are both mxl so $\mathfrak{p} = \mathfrak{m}$

$\Rightarrow \{\mathfrak{m}_1, \ldots, \mathfrak{m}_e\} = \mathrm{MaxSpec}(R)$ (set of mxl ideals of R)

• **Claim 1:** $\mathcal{N} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} = \bigcap_{\mathfrak{p} \text{ min prime}} \mathfrak{p} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell \underset{\substack{\downarrow \\ \text{pairwise coprime}}}{=} \mathfrak{m}_1 \cdots \mathfrak{m}_\ell$

• **Claim 2:** $\exists n$ st $\mathcal{N}^n = (0)$

Pf/ $\mathcal{N}$ is f.g., so $\mathcal{N} = (x_1, \ldots, x_s)$ & $x_1^{m_1} = \cdots = x_s^{m_s} = 0$
$m_1, \ldots, m_s \in \mathbb{Z}_{\geq 1}$.

Pick $k = \max\{m_1, \ldots, m_s\}$ so $x_i^k = 0 \ \forall i = 1, \ldots$

Pick $n > s(k-1)$

Write $y_i = \sum_{j=1} a_j^{(i)} x_j \in \mathcal{N}$

$\Rightarrow y_1 \cdots y_n = (a_1^{(1)} x_1 + \cdots + a_\ell^{(1)} x_\ell) \cdots (a_1^{( )} x_1 + \cdots + a_\ell^{(n)} x_\ell)$ equals 0 since after distributing, each summand must contain some x_j raised to a power $> k-1$ (ie $\geq k$). $\square$

• Using the claim & the proof of the Structure Theorem, we have $R \simeq R/\mathfrak{m}_1^n \times \cdots \times R/\mathfrak{m}_\ell^n$

Each $R/\mathfrak{m}_j^n$ is Noetherian, of dim 0 & local (with unique maximal ideal $\overline{\mathfrak{m}}_j = \mathfrak{m}_j/\mathfrak{m}_j^n$).

The Noetherian condition says $\overline{\mathfrak{m}}_j^{\,i}/\overline{\mathfrak{m}}_{j}^{\,i+1}$ is an $R/\mathfrak{m}_j$-v.s of finite dimension (f.g as an R-module).

The proof technique of (Artinian + local $\Rightarrow$ Noetherian) works here as well. And shows $R/\mathfrak{m}^n$ is Artinian

If $\exists$ strictly descending chain of ideals in $R/\mathfrak{m}_j^n$ $\Rightarrow \exists$ infinite list of $\neq$ subspaces in $\mathfrak{m}_j^i/\mathfrak{m}_j^{i+1}$ $i=1,\dots,n$. Contr!

To finish, we need only show that finite product of Artinian rings is Artinian. (any ideal in $R_1\times\dots\times R_n$ has the form $\mathfrak{a}_1\times\dots\times\mathfrak{a}_n$ for $\mathfrak{a}_i\subseteq R_i$ ideal) $\square$

§2 More on Primary Decomposition:

- Next we discuss uniqueness properties of this decomposition

⚠ Decompositions are in general NOT unique, but certain features will be:

Ex: $\mathfrak{a}=(x^2,xy)\subseteq R=K[x,y]$ (K = any field)

Let $\mathfrak{p}_1=(x)$, $\mathfrak{p}_2=(x,y)$ Both are prime ideals

$\mathfrak{a}=\mathfrak{p}_1\cap\mathfrak{p}_2^2=\mathfrak{p}_1\cap(x^2,y)$ are 2 distinct primary dec of $\mathfrak{a}$

- $\mathfrak{p}_1$ primary $fg\in(x)$ & $x\nmid g \Rightarrow x\mid f$.
- $\mathfrak{p}_2^2$ " because $r(\mathfrak{p}_2^2)=(x,y)$ is max'l.
- (x^2,y) " " $r((x^2,y))=(x,y)$ ——.

Both have $\mathfrak{p}_1$ in common. This is no accident!

- From now on, we write $\boxed{\mathfrak{a}=\mathfrak{q}_1\cap\dots\cap\mathfrak{q}_\ell}$ (*) primary decom

$\mathfrak{p}_i=r(\mathfrak{q}_i)$ (prime) for all $i=1,\dots,\ell$

STEP 1: Create a "reduced" primary decomp from (*)

Def: The decomposition (*) is **reduced** if the following 2 assumptions hold:

(1) $\mathfrak{P}_1, \ldots, \mathfrak{P}_\ell$ are all distinct.

(2) $\mathfrak{q}_i \not\supseteq \bigcap_{\substack{1\leq j\leq \ell \\ j\neq i}} \mathfrak{q}_j$ for all $i=1,\ldots,\ell$. (i.e. no $\mathfrak{q}_i$ is redundant)

Removing redundant $\mathfrak{q}_i$'s we can assume (2) holds.

Our next lemma says that (1) can always be achieved:

Lemma: If $\tilde{\mathfrak{q}}_1, \ldots, \tilde{\mathfrak{q}}_n$ are primary ideals with $r(\tilde{\mathfrak{q}}_i) = \mathfrak{P}$ $\forall i=1,\ldots,n$, then $\tilde{\mathfrak{q}} = \bigcap_{i=1}^n \mathfrak{q}_i$ is also primary and $r(\tilde{\mathfrak{q}}) = \mathfrak{P}$.

Proof: • $r(\tilde{\mathfrak{q}}) = \bigcap_{i=1}^n r(\tilde{\mathfrak{q}}_i) = \bigcap_{i=1}^n \mathfrak{P} = \mathfrak{P}$

• Pick $xy \in \tilde{\mathfrak{q}}$ with $y \notin \tilde{\mathfrak{q}}$. Then, $y \notin \tilde{\mathfrak{q}}_j$ for some j & $xy \in \mathfrak{q}_j \underset{\mathfrak{q}_j \text{ primary}}{\Longrightarrow} x^n \in \mathfrak{q}_j$, i.e. $x \in r(\mathfrak{q}_j) = \mathfrak{P} = r(\tilde{\mathfrak{q}})$ so $x^N \in \tilde{\mathfrak{q}}$ for some $N>0$, as we wanted. So $\tilde{\mathfrak{q}}$ is primary $\square$

STEP 2: Analyze uniqueness features of reduced prim. decomp.

Theorem 2 The set of prime ideals $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_\ell\}$ is uniquely determined by $\mathfrak{a}$. More precisely:

$$\{\mathfrak{P}_1, \ldots, \mathfrak{P}_\ell\} = \{ r((\mathfrak{a}:x)) : x \in R \ \& \ r((\mathfrak{a}:x)) \text{ is prime}\}$$

this does NOT require a primary decomp.
(so (LHS) is indep of our choice of red. primary decomp)

To prove this statement, we need some technical lemmas:

Lemma: Let $\mathfrak{q} \subsetneq R$ be primary & $\mathfrak{P} := r(\mathfrak{q})$. Given $x \in R$, we have:

(1) $x \in \mathfrak{q} \implies (\mathfrak{q} : x) = R$

(2) $x \notin \mathfrak{q} \implies (\mathfrak{q} : x)$ is primary & $r(\mathfrak{q} : x) = \mathfrak{P}$.

(3) $x \notin \mathfrak{P} \implies (\mathfrak{q} : x) = \mathfrak{q}$.

Recall: $(\mathfrak{a} : \mathfrak{b}) = \{ r \in R : r\mathfrak{b} \subseteq \mathfrak{a} \}$ for $\mathfrak{a}, \mathfrak{b}$ ideals.

Proof: (1) is clear, For (3): if $y \in (\mathfrak{q} : x)$ & $x \notin \mathfrak{P}$, then $y \in \mathfrak{q}$ (otherwise, $xy \in \mathfrak{q}$ & $y \notin \mathfrak{q} \implies x \in r(\mathfrak{q}) = \mathfrak{P}$)

So $\underset{\substack{\uparrow \\ \text{Always}}}{\mathfrak{q} \subseteq (\mathfrak{q} : x)} \subseteq \mathfrak{q}$ gives $(\mathfrak{q} : x) = \mathfrak{q}$.

For (2): We begin by proving $(\mathfrak{q} : x) \subset \mathfrak{P}$. Let $y \in (\mathfrak{q} : x)$ so $xy \in \mathfrak{q}$. Since $x \notin \mathfrak{q}$, then $y^n \in \mathfrak{q}$ for some $n > 0$, ie, $y \in \mathfrak{P}$.

So $\mathfrak{q} \subseteq (\mathfrak{q} : x) \subset \mathfrak{P}$. Taking radicals gives

$$\mathfrak{P} = r(\mathfrak{q}) \subseteq r((\mathfrak{q} : x)) \subseteq r(\mathfrak{P}) = \mathfrak{P} \implies r((\mathfrak{q} : x)) = \mathfrak{P}.$$

- To finish, we show $(\mathfrak{q} : x)$ is primary Pick $yz \in (\mathfrak{q} : x)$, ie $yzx \in \mathfrak{q}$. We'll use the contrapositive in the definition of primary ideal. Assume $y^n \notin (\mathfrak{q} : x)$ $\forall n > 0$. Then, $y \notin \mathfrak{P}$ & so $y^n \notin \mathfrak{q}$ $\forall n > 0$).

Then, $xz \in \mathfrak{q}$ so $z \in (\mathfrak{q} : x)$. $\square$