Lecture 27: Modules over PIDs

Recall: R PID = • all ideals can be generated by just 1 element
• R a commutative domain (no zero divisors)

Today's Goal: Classify finitely generated modules over PID's

Application: finitely generated abelian gps = fg $\mathbb{Z}$-modules
$\rightsquigarrow$ Classification for f.g ab. gps.

• Elements of a module M come in 2 flavours:
- $\mathrm{Ann}(m) = (0)$ $\rightsquigarrow$ m is a "free" element
- $\mathrm{Ann}(m) \neq (0)$ $\rightsquigarrow$ $\mathrm{Ann}(m) = (f)$ $f \neq 0$
  $\underset{R}{\cap\!|}$ ideal — So m is a "torsion element".

$\rightsquigarrow$ M will be decomposed into a "free part" and a "torsion part".

§1. Free modules:

Def: An R-module M is free if $M \underset{\varphi}{\xleftarrow{\simeq}} \bigoplus_{i \in I} R \quad (= R^{\oplus I})$ for some I.

We say $\{ \varphi(e_i) : i \in I \}$ is a basis for M.

Theorem: If R is commutative an M is a free module, then any two bases for M have the same cardinality. (We call it the rank of M)

Pf/ Consider $\mathfrak{m}$ a maximal ideal on R. Then $\overline{M} = M/\mathfrak{m}M$ is a k vector space for $k = R/\mathfrak{m}$. Then, $\overline{M}$ has a basis & all bases of $\overline{M}$ have the same cardinality.

Furthermore: If $(x_i)_{i\in I}$ is a basis for M, then $\overline{x_i} = x_i + mM$ is a basis for $\overline{M}$ (so $|I| = \dim_k \overline{M}$ does not depend on the basis)

Indeed, we show $\{\overline{x_i}\}_{i\in I}$ both spans & is li:

① $\{\overline{x_i}\}_{i\in I}$ spans:

If $\overline{x} \in M/mM$, then $x = \sum_{\substack{i\in I \\ \text{finite}}} a_i x_i$ with $a_i \in R$

so $\overline{x} = \sum_{\substack{i\in I \\ \text{finite}}} (a_i + m)\overline{x_i}$

② $\{\overline{x_i}\}_{i\in I}$ is li: If $\overline{0} = \sum_{\substack{i\in I \\ \text{finite}}} (a_i+m)\overline{x_i}$, then

$\sum_{\substack{i\in I \\ \text{finite}}} a_i x_i \in mM$, so $x = \sum_{\substack{i\in I \\ \text{finite}}} a_i x_i = \sum_{j=1}^{m} b_j y_j$ with $y_j \in M$, $b_j \in m$

With $y_j = \sum_{\substack{i\in I \\ \text{finite}}} c_i^{(j)} x_i$

$\rightsquigarrow \sum_{j=1}^{m} b_j \sum_{\substack{i\in I \\ \text{finite}}} c_i^{(j)} x_i \underset{\text{Assoc}}{=} \sum_{\substack{i\in I \\ \text{finite}}} \underbrace{\left(\sum_{j=1}^{m} b_j c_i^{(j)}\right)}_{\in m} x_i$

By definition of $\bigoplus_{i\in I} R$, $a_i \in m$ $\forall i$ in supp of x, $a_i = 0$ otherwise.

So $a_i + m = 0$ $\forall i$. We conclude $\{\overline{x_i}\}$ is l.i. □

- Next, we need to ensure freeness is preserved for submodules:

This is not true in general!

Example: $R = K[x,y]$ $\quad M = R$ is free of rank 1. but
$I = (x, y)$ is not a free submodule

- $I \not\cong R$ (not a cyclic module)
- We have the obvious relation: $\underline{y}\underline{x} - \underline{x}\underline{y} = 0$ (in I)
- Any $\{f_i\}_{i \in I}$ generating set will have obvious relation $f_i f_j - f_j f_i = 0$ (in I).

Theorem 2: Let F be a free module over a PID R & M a submodule. Then, M is free and $\text{rank}(M) \leq \text{rank}(F)$.

Proof: We discuss the finite case (For the case when rank (F) is infinite, see HW10). Assume F has a basis $\{x_i\}_{i=1}^n$ $\quad (n = \text{rank } F)$.

Let $M_r = M \cap (x_1, \ldots, x_r)$ for $r = 1, \ldots, n$.

We show M_r is free of rank $\leq r$ by induction on r:

- **Base case**: $r = 1$ $\quad M_1 = M \cap (x_1)$ is a submodule of (x_1), so $M_1 = (a_1 x_1)$ for some $a_1 \in R$.
  So $M_1 = 0$ or free with $M_1 \simeq R$ because $\text{Ann}(x_1) = 0$ & R is a domain.
- **Inductive step**: Consider the following set of R

$$\mathfrak{a} = \{a \in R : \exists\, x \in M \text{ with } x = b_1 x_1 + \cdots + b_r x_r + a x_{r+1} \text{ for } b_1, \ldots, b_r \in R\}$$

Claim: $\mathfrak{a}$ is an ideal (because M is an R-module)

Since R is a PID, then $\mathfrak{a} = (a_{r+1})$ for some $a_{r+1} \in R$.

Two cases:

(1) If $a_{r+1} = 0$, then $M_{r+1} = M_r$ so M_{r+1} is free of rank $\leq r$.

(2) If $a_{r+1} \neq 0$, we pick $w \in M_{r+1}$ with $w = \underbrace{b_1 x_1 + \cdots + b_r x_r}_{\in (x_1, \ldots, x_r)} + a_{r+1} x_{r+1}$

For any $x \in M_{r+1}$ we write $x = a_1 x_1 + \cdots + a_r x_r + \underbrace{(c \cdot a_{r+1})}_{\in \mathfrak{a}} x_{r+1}$

so $x - cw \in M \cap (x_1, \ldots, x_r)$

So $\boxed{M_{r+1} = M_r + (w)}$

Clearly, $M_r \cap (w) = (0)$

$$M_{r+1} = M_r \oplus (w), \quad M_r \cong R^s \ (s \leq r), \quad (w) \overset{(*)}{\cong} R$$

(*) $\operatorname{Ann}(w) = (0)$ because $a_{r+1} \neq 0$.

□

Corollary: If E is a f.g module over a PID (R) & E' is a submodule, then E' is f.gen.

Pf/ View $E = R^n/\text{rels}$. $\varphi: R^n \twoheadrightarrow R^n/\text{rels} = E$

Then $M = \varphi^{-1}(E') \subseteq R^n$ so free of rank $\leq n$. If $B = \{x_1, \ldots x_s\}$ is a basis for M, then $E' = \varphi(M) = \langle \varphi(x_1), \ldots \varphi(x_s) \rangle$ is fin. gen. R-module. □

Alternative Proof: Use PID $\Rightarrow$ Noetherian

E f.g module & R Noeth, so E is Noetherian as a module. □

§ 2. Torsion for modules:

Def: Let M be an R-module. We say M is a **torsion module** if given $x \in M$ $\exists\, a \in R \setminus \{0\}$ with $ax = 0$ (equivalently, $\operatorname{Ann}(x) \neq (0)$ $\forall x \in M$).

Obs: Finite abelian gp translates to finitely generated torsion module

Def: A **Torsion element** x of a module M is an element with $\mathrm{Ann}(x) \neq (0)$. Write $\boxed{M_{tor}} = \{\text{torsion elements of } M\}$

Def If $M_{tor} = \{0\}$, we say M is **Torsion free**.

⚠ Torsion free + fg $\not\Rightarrow$ Free

Ex: $M = (x,y)$ torsion free $\mathbb{K}[x,y]$-mod, but NOT free.

However, the statement is true for modules over PIDs:

Prop: M a fg module over a PID $= R$. If M is torsion free, then M is free.

Pf/ Consider $S = \{v_1, \ldots, v_n\}$ a maximal li set of elements of M among a set $\mathcal{Y} = \{y_1, \ldots, y_m\}$ of generators of M.

(Here: li means $a_1 v_1 + \cdots + a_n v_n = 0$, $a_i \in R \Rightarrow a_i = 0 \; \forall i$)

- Pick $y \in \mathcal{Y} \setminus S$. Then: $\exists\, a, b_1, \ldots, b_n \in R$, not all 0, so

$$ay + b_1 v_1 + \cdots + b_n v_n = 0.$$

Since S is li, then $a \neq 0$.

- If $y \in S$, then $y = v_i$ & $\overset{a}{1} \cdot y - 1 \cdot v_1 = 0$.

Conclusion: For all $j = 1, \ldots m$ we can find $a_j \in R \setminus \{0\}$ with $a_j y_i \in (v_1, \ldots, v_n)$. Take $a = a_1 \cdots a_m$.

Then $aM \subseteq (v_1, \ldots, v_n)$ & $a \neq 0$. (R domain.)

We take the multiplication map $\varphi_a : M \longrightarrow (v_1, \ldots, v_n)$
$$m \longmapsto am.$$

- φ_a is injective because M is torsion free.

- $N = (v_1, \ldots, v_n)$ is a free module $(\cong R^n)$, so $\varphi_a(M) \subset N$ is also free (by Thm 2).

Conclude: $M \cong \varphi_a(M)$ is free of rank $\leq$ rank $N = n \leq m$ = # gens of M.

$\Rightarrow$ rank $M \leq \min \{\#(\text{gen set}) \text{ for } M\}$. □

The previous proposition allow us to decompose f.g modules over PID's as a direct sum of a torsion & a free-module.

Theorem 3: Fix R a PID and M a f.g R-module. Then, M/M_{tor} is a free R-module. Furthermore, there exists a free submodule F of M with $M = M_{tor} \oplus F$.

The rank of F is uniquely determined by M.

Proof: Next Time.