Lecture 28: Modules over PID's II

Recall: • Last time we talked about free modules over a PID R

$$M \cong \bigoplus_{i \in I} R \quad \text{via a basis } \{e_i\}_{i \in I} \quad (\text{generates} + LI/R)$$

• Defined Torsion elements: $x \in M$ with $\mathrm{Ann}(x) \neq (0)$

$M_{Tor} = \{$ torsion elements $\}$ submodule of M

• M is Torsion free module $\Leftrightarrow M_{Tor} = \{0\}$

Theorem 1: Size of the basis is unique $= \mathrm{rank}(M)$

Theorem 2: F free module over PID & M submodule, then M is free & $\mathrm{rank}(M) \leq \mathrm{rank}(F)$

Torsion free + fg $\not\Rightarrow$ free for general R

Proposition: M is fg over a PID & M is torsion free $\Rightarrow$ free.

§1 Splitting of fg modules over PIDs:

Theorem 3: Fix R a PID and M a f.g R-module. Then, M/M_{Tor} is a free R-module. Furthermore, there exists a free submodule F of M with $M = M_{Tor} \oplus F$.

The rank of F is uniquely determined by M.

Proof: • We first show that $\overline{M} = M/M_{Tor}$ is Torsion free.

Let $\overline{x} \in \overline{M}$ and $b \in R$ with $b\overline{x} = 0$ in $\overline{M}$. Then $bx \in M_{Tor}$ so $\mathrm{Ann}(bx) \neq (0)$. But $\mathrm{Ann}(bx) = (c)$ $c \neq 0$

gives $(bc)x = 0$ in M.

So either $bc = 0$ or $x \in M_{Tor}$ ($\Rightarrow \bar{x} = 0$)

$\Downarrow c \neq 0$

$b = 0$. & $\bar{x}$ is not a torsion element of $\bar{M}$.

- M is fg, so $\bar{M}$ is fg

- $\bar{M}$ is fg & torsion free. By Proposition, it is free as an R-module. It's rank is uniquely determined by M.

- To find F, we need a lemma applied to $\varphi: M \twoheadrightarrow M/M_{Tor}$

<u>Lemma</u>: Consider M & M' two modules over a PID R. Assume M' is free & let $f: M \to M'$ be a surjective homomorphism of R-modules. Then, there exists a free submodule N of M such that

(1) $f|_N$ induces an isomorphism $f|_N : N \xrightarrow{\sim} M'$.

(2) $M = N \oplus \operatorname{Ker} f$.

<u>Proof</u>. Pick a basis $B = \{x_i'\}_{i \in I}$ for M'. For each i, let

$x_i \in M$ with $f(x_i) = x_i'$.

Take $N = (x_i : i \in I)$

<u>Claim</u>: $\{x_i : i \in I\}$ is li

Pf/ $\sum_{\substack{i \in I \\ \text{finite}}} a_i x_i = 0 \rightsquigarrow \sum_{\substack{i \in I \\ \text{finite}}} a_i \underbrace{f(x_i)}_{= x_i'} = 0 \Rightarrow \underset{B \text{ basis}}{a_i = 0} \; \forall i$.

<u>Conclusion</u>: N is free with basis $\{x_i\}_{i \in I}$.

- Clearly: $f|_N : N \twoheadrightarrow M'$

- For $x \in M$, we can find $a_i \in R$ (finitely many $\neq 0$) with $f(x) = \sum_{\substack{i \in I \\ \text{finite}}} a_i x_i' = \sum_{\substack{i \in I \\ \text{finite}}} a_i f(x_i)$

So $x - \sum_{\substack{i \in I \\ \text{finite}}} a_i x_i \in \operatorname{Ker} f$. Thus, $M = N + \operatorname{Ker} f$

- $N \cap \operatorname{Ker} f = (0)$ because $\{x_i'\}_{i \in I}$ is a basis.

So $f|_N : N \xrightarrow{\sim} M'$ & $M = N \oplus \operatorname{Ker} f$. $\square$

§2 Classification of modules over PID's.

Fix R a PID

Def: We say $p \in R$ is a prime element if $(0) \neq (p)$ is a prime ideal of R.

- We select representatives for the prime elements of R, modulo units.

Examples: $\mathbb{Z} \rightsquigarrow$ positive prime numbers

$\mathbb{K}[x] \rightsquigarrow$ monic irreducible polynomials $(LT(f) = 1)$

Def An Exponent of M = an element in $\operatorname{Ann}(M) \setminus \{0\}$

Notation: We say $x \in M$ is a p-torsion point if $p^n \cdot x = 0$ for some $n \geq 1$. (equiv., $\exp(x)$ is a positive power of p)

Def: Given $a \in R \setminus \{0\}$, we write $\boxed{M_a} = \operatorname{Ker}\left(M \xrightarrow{a\cdot} M\right)$, $x \longmapsto a \cdot x$

Def: An R-module M is cyclic if $M \simeq R/(a)$ for some a.

Obs: If $a \neq 0$, we can write $a = u\, p_{i_1}^{n_1} \cdots p_{i_r}^{n_r}$ — p_i prime reps, $n_i \in \mathbb{Z}_{\geq 0}$, $u \in R^\times$

(use primary decomp of (a))

Def: A p-module M ($M = M_{p^n}$ for some n) is of type $(p^{r_1}, \ldots, p^{r_s})$ if it is isomorphic to $\prod_{i=1}^{s} R/(p_i^{r_i})$

If p is understood, we say M has type $(r_1, \ldots, r_s)$

- M is a torsion module if $M = M_{tor}$.

Classification Theorem I: If $(0) \neq M$ is a fg torsion module over a PID R, then: $M = \bigoplus_{p_i \text{ prime}} M_{p_i^{n_i}}$ for suitable $n_i \in \mathbb{Z}_{\geq 0}$ with $M_{p_i^{n_i}} \neq \{0\}$.

Furthermore: $M_{p_i^{n_i}} \cong R/(p^{\nu_1^{(i)}}) \oplus \cdots \oplus R/(p^{\nu_s^{(i)}})$

with $n_i = \nu_1^{(i)} \geq \nu_2 \geq \cdots \geq \nu_s \geq 1$ & the sequence (ν_i) is uniquely determined by M & p.

Proof: First we discuss the decomposition of M as a direct sum of p-torsion modules.

Claim 0: M has a nonzero exponent a.

Pf/ Write $M = (x_1, \ldots, x_n)$. We know $\mathrm{Ann}(x_i) = (a_i) \neq (0)$ because M is a torsion module.

Take $a = a_1 \cdots a_n \neq 0$ then $aM = \{0\}$. □

- Since $M \neq \{0\}$, $a \notin R^\times$ so $\mathrm{Ann}(M) \neq (0), R$.

WLOG, assume $\boxed{\mathrm{Ann}(M) = (a)}$, so $M = M_a$.

- Assume $a = bc$ with $(b, c) = 1$. Pick x, y with $1 = xb + yc$

<u>Claim 1</u>: $M = M_b \oplus M_c$

Pf/. $M_b \cap M_c = \{0\}$ since $m \in M_b \cap M_c$ forces $\begin{matrix} bm = 0 \\ cm = 0 \end{matrix}$ so $1 \cdot m = (xb + yc) \cdot m = 0 + 0 = 0$

• $(\subseteq)$ Pick $m \in M$. Then $m = (xb+yc)m$

$$= \underbrace{xbm}_{=m_1} + \underbrace{ycm}_{=m_2} \in M_c + M_b$$

$cm_1 = x\underbrace{cb}_{=a}m = x \cdot 0 = 0$ so $m_1 \in M_c$

$bm_2 = y\underbrace{bc}_{=a}m = y \cdot 0 = 0$ so $m_2 \in M_b$ $\square$

• We <u>factor</u> a as: $a = u\, p_{i_1}^{n_{i_1}} \cdots p_{i_r}^{n_r}$ $\quad u \in R^\times$, p_{i_j} primes, $n_{i_j} > 0$.
(via Primary decomp of (a)).

<u>CASE 1</u>: $r = 1$

Then $a = u p^n$ & $M = M_a = M_{p^n}$

<u>CASE 2</u>: $r > 1$

Then $a = bc$ with $b = u\, p_{i_1}^{n_{i_1}}$ & $c = p_{i_2}^{n_{i_2}} \cdots p_{i_r}^{n_{i_r}}$

<u>Claim 2</u>: $(b,c) = 1$

Pf/ $(b,c) = (d)$ (R is a PID) so $b = dx$, $c = dy$

But $b = u\, p_{i_1}^{n_{i_1}} = dx$ forces $d = w\, p_{i_1}^{s_{i_1}}$ with $0 \leq s_{i_1} \leq n_{i_1}$, $w \in R^\times$
(primary decomp are unique for PIDs)

Also $c = p_{i_2}^{n_{i_2}} \cdots p_{i_r}^{n_{i_r}} = dy$ forces $d = v\, p_{i_2}^{s_{i_2}} \cdots p_{i_r}^{s_{i_r}}$ $\quad v \in R^\times$, $s_{i_j} \geq 0$

But $p_{i_1}^{s_{i_1}}, p_{i_2}^{s_{i_2}} \cdots p_{i_r}^{s_{i_2}}$ are coprime, so only option is $s_{i_j} = 0 \;\forall j$
ie $d \in R^\times$.

By our Claim 1 $M = M_b \oplus M_c$.

- From Cases 1 & 2, we induct on the number of prime factors in b & c to show

$$M = M_{p_{i_1}^{n_{i_1}}} \oplus \cdots \oplus M_{p_{i_r}^{n_{i_r}}}$$

Next, we show that $M_{p_{i_j}^{n_{ij}}}$ admits the claimed decomposition.

It suffices to focus on p-torsion modules. Uniqueness will be proven at the end.

- Assume $M = M_{p^n}$ with n minimal. Notice $\overline{M} = M/_{pM}$ is a k-vsp with $k = R/_{(p)}$ (in a PID prime & $\neq 0$ $\Rightarrow$ mxl)
- Since M is fg: $\dim_k \overline{M} < \infty$
- We argue by induction on $\dim_k \overline{M}$, using the following lemma for the inductive step

Lemma: Assume $\mathrm{Ann}(M) = (p^n)$ & pick $x \in M$ with $\mathrm{Ann}(x) = (p^n)$. Consider the ses

$$0 \longrightarrow (x) \longrightarrow M \xrightarrow{\pi} \underset{\text{"} M/_{(x)}}{N} \longrightarrow 0$$

Then (1) $\dim_k N/_{pN} < \dim_k M/_{pM}$

(2) π admits a section. (assuming N decomposes as expected)

Pf/ Next Time.

We will complete the proof of the Classification Thm in the next lecture.