Lecture 29: Modules over PID III

Recall: • Last time we talked about free modules over a PID R

$M \cong \bigoplus_{i\in I} R$ via a basis $\{e_i\}_{i\in I}$ (generates + LI/R)

• Defined <u>Torsion elements</u>: $x \in M$ with $\mathrm{Ann}(x) \neq (0)$

$M_{tor} = \{$ torsion elements $\}$ submodule of M

• M is <u>Torsion free</u> module $\Leftrightarrow M_{tor} = \{0\}$

<u>Thm 1</u>: $M \neq 0$ fg module over a PID

$$M \cong M_{tor} \oplus F \qquad F \text{ free } \cong M/M_{tor}$$

§1 Classification v.1:

• Pick representatives for <u>prime elements</u> in R $\{p_i\}_{i\in I}$
$\hookrightarrow (p) \neq 0$ prime idea

<u>**Classification Theorem 1**</u>: If $(0) \neq M$ is a fg Torsion module over a PID R, then:
$$M = \bigoplus_{p_i \text{ prime}} M_{p_i^{n_i}}$$
for suitable $n_i \in \mathbb{Z}_{\geq 0}$ with $M_{p_i^{n_i}} \neq \{0\}$.

Furthermore: $M_{p_i^{n_i}} \cong R/(p^{v_1^{(i)}}) \oplus \cdots \oplus R/(p^{v_s^{(i)}})$

with $n_i = v_1^{(i)} \geq v_2 \geq \cdots \geq v_s \geq 1$ & the sequence (v_i) is uniquely determined by M & p.

<u>Last time</u>: Wrote $\mathrm{Ann}(M) = (a)$ $a \neq 0$, $a \notin R^{\times}$.

$a = u\, p_{i_1}^{n_{i_1}} \cdots p_{i_r}^{n_{i_r}} \rightsquigarrow M = M_a = \bigoplus_{j=1}^{r} M_{p_{i_j}^{n_{i_j}}}$

$u \in R^{\times}$ p_{i_j} prime reps $n_{i_j} \geq 1$

- Uniqueness of p_{ij} & n_{ij} follows from uniqueness of primary decomposition for PIDs.

<u>TODAY</u>: We focus on the classification of p-torsion modules

Uniqueness will be proven at the end.

- Assume $M = M_{p^n}$ with n minimal. Notice $\overline{M} = M/pM$ is a k-vsp with $k = R/(p)$ (in a PID prime & $\neq 0$ $\Rightarrow$ mxl)
- Since M is fg: $\dim_k \overline{M} < \infty$

- We argue by induction on $\dim_k \overline{M}$, using the following lemma for the inductive step

<u>Lemma</u>: Assume $\mathrm{Ann}(M) = (p^n)$ & pick $x \in M$ with $\mathrm{Ann}(x) = (p^n)$

Consider the ses $0 \longrightarrow (x) \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$, $N = M/(x)$

Then (1) $\dim_k N/pN < \dim_k M/pM$

(2) π admits a section. (assuming N decomposes as expected)

Pf/ (1)
$$\begin{tikzcd} M \arrow[r, two heads, "\pi_1"] \arrow[rr, bend right, "f"'] \arrow[d] & N \arrow[r, two heads, "\pi_2"] & N/pN \\ M/pM \arrow[urr, "\overline{F}"'] & & \end{tikzcd}$$

$\pi_1(pM) = p\pi_1(M) = pN$

so $pM \subseteq \ker f$

we get $\overline{F}: M/pM \twoheadrightarrow N/pN$

- $\overline{F}$ is k-linear map
- $\overline{F}$ is surjective so $\dim_k N/pN \leq \dim_k M/pM < \infty$

- $(x) \in M$ satisfies $\overline{F}(x + pM) = 0$. & $x + pM \neq pM$ because $p^{n-1} x \neq 0$.

• We get $R/(p) \cong k \cong \frac{x}{p(x)} \hookrightarrow \frac{M}{pM} \xrightarrow{\bar{F}} \frac{N}{pN}$ & $\bar{F}\big|_{\frac{x}{p(x)}} = 0$

<u>Conclusion</u> $\dim_k \frac{N}{pN} < \dim_k \frac{M}{pM}$.

(2) We assume $N \overset{\varphi}{\cong} \bigoplus_{i=1}^{s} R/(p^{\nu_i})$ with $\nu_1 \geq \cdots \geq \nu_s$

Consider $\{\bar{y}_1, \dots, \bar{y}_s\}$ where $\varphi(y_i) = e_i$. $\mathrm{Ann}(\bar{y}_i) = p^{\nu_i}$.

We want to lift each y_i to M so that $\begin{cases} \cdot\ \mathrm{Ann}(y_i) = \mathrm{Ann}(\bar{y}_i) \\ \cdot\ y_i + R(x) = \bar{y}_i \in N \end{cases}$

• It suffices to do this for a single $\bar{y} \in N \setminus \{0\}$

Assume $\mathrm{Ann}(\bar{y}) = (p^l)$ for some $l \geq 1$. Pick an $y \in M$ with $y + R(x) = \bar{y}$.

Then $p^l y \in R(x)$. Write $p^l y = bx$ for $b \in R$ & factor b as $b = p^s c$ with $p \nmid c$ & $s \geq 0$.

• Since $p^n x = 0$ we may assume $\boxed{s \leq n}$ (otherwise, $p^s c x = 0 = p^n c x$)

• If $s = n$, then $p^l y = 0$
$p^{l-1} y \notin R(x)$ so $p^{l-1} y \neq 0$ $\Big\}$ $\mathrm{Ann}(y) = \mathrm{Ann}(\bar{y})$

• If $s < n$, then $\mathrm{Ann}(p^s c x) = (p^{n-s})$ so $\mathrm{Ann}(y) = (p^{l+n-s})$

Since $p^n y = 0$ we get $l + n - s \leq n$, ie $\boxed{l \leq s}$

So $\boxed{y' = y - p^{s-l} c x}$ satisfies

• $y' + R(x) = \bar{y}$

&

• $\mathrm{Ann}(y') = (p^l) = \mathrm{Ann}(\bar{y})$.

$\left(p^l y' = p^l y - p^s c x = 0 \ \& \ p^{l-1} y' = p^{l-1} y - p^{s-1} c x = 0 \text{ forces } p^{l-1} y \in R(x) \right)$

Contr.!

- Assume we've lifted $\bar{y}_1, \dots, \bar{y}_s$ to $y_1, \dots, y_s$ with $\mathrm{Ann}(y_i) = \mathrm{Ann}(\bar{y}_i)$ & $y_i + R(x) = \bar{y}_i \in N$

Then $M = R(x) \oplus M'$ where $M' = R(y_1 \cdots y_s)$

Since
- $M' \cap R(x) = \{0\}$
- $M/R(x) = M' \simeq N$ $\square$

<u>End of the proof of Classification Thm.</u> (existence of the decomposition)

We proceed by induction on $\dim_k M/pM$ ($k = R/(p)$ field)

- M f.g. so $\dim_k M/pM < \infty$. Here $M = M_{p^n}$ n minimal $n \geq 1$.

<u>Base case</u>: $\dim_k M/pM = 1 = \dim_k (x)/p(x)$ forces $M = (x)$ so $M \simeq R/(p^n)$.

<u>Inductive step</u>: Assume N admits a decomp. since $\dim_k N/pN < \dim_k M/pM$.

Using the section to $M \longrightarrow N$ from the proof of the lemma we

We build $y_1, \dots, y_s$ from the decomp of N.

- Since $N = \bigoplus_{i=1}^{s} R(\bar{y}_i) \simeq \bigoplus_{i=1}^{s} R/(p^{\nu_i})R$ $\nu_1 \geq \dots \geq \nu_s$

and $(p^n) = \mathrm{Ann}(M) \subseteq \mathrm{Ann}(N') = (p^{\nu_1})$ we get $n \geq \nu_1 \geq \dots \geq \nu_s$.

- To finish, we show $R(y_1 \cdots y_s) = R(y_1) \oplus \cdots \oplus R(y_s)$

Indeed if $a_1 y_1 + \cdots + a_s y_s = 0$, we want to show $a_i y_i = 0$ $\forall i$

Viewed in N: $a_1 \bar{y}_1 + \dots + a_s \bar{y}_s = 0$ in $N = \bigoplus_{i=1}^{s} R(\bar{y}_i)$ forces

$a_i \bar{y}_i = 0$ so $a_i \in \mathrm{Ann}(\bar{y}_i) = \mathrm{Ann}(y_i) \Rightarrow a_i y_i = 0$

Thus $M = \underset{R/(p^n)}{\underset{\wr\wr}{(x)}} \oplus \underset{R/(p^{\nu_1})}{\underset{\wr\wr}{R(y_1)}} \oplus \cdots \oplus \underset{R/(p^{\nu_s})}{\underset{\wr\wr}{R(y_s)}}$ $n \geq \nu_1 \geq \dots \geq \nu_s$ $\square$

Uniqueness Proof: $M_{p^n} = (x) \oplus \bigoplus_{i=1}^{s} R(y_i)$ $n \geq \nu_1 \geq \dots$

For M_{p^n} n = exponent of $\mathrm{Ann}(M_{p^n}) = (p^n)$ $\mathrm{Ann}(x) = (p^n)$

$\mathrm{Ann}(M_{p^n}) = (p^n)$ $\nu_1 =$ ______ $(M_{p^n}/(x)) = (p^{\nu_1})$

$\nu_2 =$ ______ $(M_{p^n}/(x, y_1)) = (p^{\nu_2})$

Claim: $s = \dim_k M/pM - 1$ $\left(k = R/(p)\right)$

Pf/ Since: $M = R(x) \oplus R(y_1) \oplus \dots \oplus R(y_s)$

Then $pM = pR(x) \oplus pR(y_1) \oplus \dots \oplus pR(y_t)$

with $\nu_{t+1} = \dots = \nu_s = 1$. (could have $t = s$)

So $\underset{\substack{\downarrow \\ k \text{ v.s.}}}{M/pM \simeq} \underbrace{\dfrac{R(x)}{pR(x)}}_{\cong k} \oplus \underbrace{\dfrac{R(y_1)}{pR(y_1)}}_{\cong k} \oplus \dots \oplus \underbrace{\dfrac{R(y_t)}{pR(y_t)}}_{\cong k} \oplus \bigoplus_{j=t+1}^{s} \underbrace{R(y_j)}_{\cong k}$

$\dim_k M/pM = 1 + t + (s - t) = s + 1$

- So # of Terms is unique!
- Assume $M = (x') \oplus \bigoplus_{i=1}^{s} R(y'_i)$ $\underset{n}{\mathrm{ord}(x')} \geq \underset{\nu'_1}{\mathrm{ord}(y'_1)} \geq \dots \geq \underset{\nu'_s}{\mathrm{ord}(y'_s)}$

But $\mathrm{ord}(x') = n = \mathrm{ord}(x)$ meaning $\mathrm{Ann}(x) = \mathrm{Ann}(x') = \mathrm{Ann}(M)$

- Next, we consider the multiplication map $p: M \xrightarrow{p\cdot} M$ & we argue by induction on n where $\mathrm{Ann}(M) = (p^n)$.

. $M' = pM$ has $\mathrm{Ann}(M') = (p^{n-1})$

- If $n = 1$, then $pM = 0$ so $\nu_1 = \nu_2 = \dots = \nu_s = 1$

$\nu'_1 = \nu'_2 = \dots = \nu'_s = 1$

- Assume $n \geq 2$ & that the decomp is unique for any module with $\text{Ann}(M) = (p^k)$ for $k \leq n-1$

By construction the decomp of M' is unique. On the other hand

$$M' \doteq pM = R(px) \oplus \bigoplus_{i=1}^{r} R(py_i) \qquad \nu_r \geq 2 \ \& \ \nu_{r+1} = \cdots = \nu_s = 1$$

$$= R(px') \oplus \bigoplus_{i=1}^{r'} R(py_i') \qquad \nu_{r'} \geq 2 \ \& \ \nu'_{r'+1} = \cdots = \nu'_s = 1$$

and $\text{Ann}(py_i) = (p^{\nu_i - 1})$, $\text{Ann}(px) = \text{Ann}(px') = (p^{n-1})$

$\text{Ann}(py_i') = (p^{\nu_i' - 1})$

Our inductive hypothesis gives: $r = r'$ &

$\nu_i - 1 = \nu_i' - 1 \quad \forall i = 1, \ldots, r$

& $\nu'_{r+1} = \cdots = \nu'_s = 1 = \nu_s = \cdots = \nu_{r+1}$

This concludes our proof.