Lecture 34: Dual Vector Spaces & Tensor products

Recall: Last time we reviewed basics on $\mathbb{K}$-vector spaces.

- Bases = generating sets + l.i

Equiv: maximal l.i subsets

- $\dim V$ = size of a basis (cardinality), well-defined HW10 P2.

Theorem: Let V be a vector space over a field $\mathbb{K}$ with $V \neq \{0\}$.

1. Let S be a l.i subset of V. Then there exists a basis B for V with $S \subset B$
2. Let Γ be a generating set for V (ie a spanning set). Then, there exists a basis B of V with $B \subset \Gamma$.

Def The dual of V, is $\boxed{V^*} = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$

1-dim'l vector space

Theorem: If V is finite-dimensional, then $\dim V^* = \dim V$.

Pf/ Let $\{v_i\}_{1\leq i\leq m}$ be a basis for V. Define $v_i^* \in V^*$ by

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \rightsquigarrow B^* = \{v_1^*, \ldots, v_m^*\} \text{ is a basis for } V^*$$

§1. Double Duals:

Remark: $W \subset V$ subspace, then V/W is a vector space

and $(V/W)^* \longleftrightarrow \{\xi \in V^* \mid \xi|_W = 0\}$

Proposition $V \xrightarrow{\varphi} (V^*)^*$ is linear & inj.

$v \longmapsto \varphi(v) : V^* \longrightarrow \mathbb{K}$

$f \longmapsto f(v)$

(alt notation for $\varphi(v)(f) = \langle v, f\rangle$

φ is surjective $\iff \dim V < \infty$.

Proof: • $\varphi(v) \in (V^*)^*$ is clear

• φ is linear ① $\varphi(v+v')(f) = f(v+v') = f(v) + f(v') = (\varphi(v) + \varphi(v'))(f)$

This is true for all f, so $\varphi(v+v') = \varphi(v) + \varphi(v')$ in $(V^*)^*$.

② $\varphi(zv)(f) = f(zv) = z f(v) = z(\varphi(v)(f)) = (z\varphi(v))(f)$

This is true for all f, so $\varphi(zv) = z \cdot \varphi(z)$ in $(V^*)^*$.

• φ is injective $(\operatorname{Ker} \varphi = \{0\})$

$\varphi(v) = 0$ means $\varphi(v)(f) = 0 \quad \forall f$.

If $v \neq 0$, then by Theorem 3 we can find a basis B for V with $v \in V$.

$\Rightarrow \exists\, v^* : V \longrightarrow 0$ with $v^*(w) = \begin{cases} 0 & w \in B \setminus \{v\} \\ 1 & w = v \end{cases}$ for $w \in B$.

Then $\varphi(v)(v^*) = v^*(v) = 1 \neq 0$ Contr!

Conclude: $v = 0$, so $\operatorname{Ker} \varphi = \{0\}$.

• We saw $V \simeq V^*$ if $\dim V < \infty$, so $V \simeq (V^*)^*$ gives the same dimension. By Rank-Nullity Theorem: φ is surjective.

• If $\dim V = \infty$ let $B = \{v_i\}_{i \in I}$ be an infinite basis for V.

Then $V = \{ \sum_{i \in I} a_i v_i \; : \; a_i = 0 \;\forall$ but finitely many $i\}$

$V^* = \{ \sum_{i \in I} a_i v_i^* \} \supsetneq F = \{ \sum_{i \in I} a_i v_i^* \; : a_i = 0$ for all but finitely many $i\}$

↳ only $=$ if $\dim V < \infty$

Claim: Any linear form $\xi : V^* \longrightarrow K$ which vanishes on F cannot be in $\operatorname{Im} \varphi$ unless $\xi = 0$.

Pf/ $F' = \{\xi \in (V^*)^* : \xi|_F = 0\} \neq \emptyset$ (complete a basis of F to a basis for V^* & take any $\xi = w^*$ for w among the added basis elements

Set $H = \{v \in V : \varphi(v)|_F = 0\}$

Then, $F' \cap \operatorname{Im}\varphi \subset \varphi(H) = 0$ so $\emptyset \neq F' \setminus 0 \subset (V^*)^* \setminus \operatorname{Im}\varphi$

$[\varphi(H)(f) = 0 \ \forall f \in F$ $\hookrightarrow \xi = \varphi(v)$ in F' so $\varphi(v)|_F = 0$ forces $v = 0$, otherwise $v^* \in F$ & $\varphi(v)(v^*) = v^*(v) = 1 \neq 0]$

$\square$

Q: How to dualize a map?

Lemma: If $f : V \to W$ is a linear map, then $f^* : W^* \to V^*$ defined by $f^*(\xi) = \xi \circ f : V \to W \to \mathbb{K}$ $\forall \xi \in W^*$ is $\mathbb{K}$-linear

More precisely: $f^*(\xi)(v) = \xi(f(v))$ $\forall v \in V, \forall \xi \in W^*$

Prop: If $\dim V = n$, $\dim W = m$ then $[f]_{B_{W^*} B_{V^*}} = [f]^T_{B_V B_W}$.
(HW 12)

§2 Bilinear Maps and Tensor Products:

Let V_1, V_2, W be 3 vector spaces over $\mathbb{K}$

Def: A bilinear map $f : V_1 \times V_2 \to W$ is a set map which is linear in each coordinate, ie:

- $\forall v_1 \in V_1$: $\quad \begin{array}{rcl} V_2 & \longrightarrow & W \\ w & \longmapsto & f(v_1, w) \end{array} \quad \in \operatorname{Hom}_{\mathbb{K}}(V_2, W)$

- $\forall v_2 \in V_2$: $\quad \begin{array}{rcl} V & \longrightarrow & W \\ w & \longmapsto & f(w, v_2) \end{array} \quad \in \operatorname{Hom}_{\mathbb{K}}(V_1, W)$

Def: The Tensor product $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space together with a bilinear map

$$\begin{array}{rcl} V_1 \times V_2 & \xrightarrow{\varphi} & V_1 \otimes_{\mathbb{K}} V_2 \\ (v_1, v_2) & \longmapsto & v_1 \otimes v_2 \end{array}$$

$\leftarrow$ indecomposable tensor

satisfying the following <u>universal property</u>: For any vector sp. W over $\mathbb{K}$ and any bilinear map $f : V_1 \times V_2 \longrightarrow W$, there exists a unique $\tilde{f} : V_1 \otimes V_2 \longrightarrow W$ linear map making the following diagram commute:

$$(*) \quad \begin{tikzcd} V_1 \times V_2 \arrow[r, "f"] \arrow[d, "\varphi"'] & W \\ V_1 \underset{\mathbb{K}}{\otimes} V_2 \arrow[ur, dashed, "\exists ! \tilde{f} \text{ linear}"'] & \end{tikzcd}$$

<u>Idea</u>: Build a space so that $\tilde{f}$ is linear.

<u>Obs</u>: The pair $(V_1 \underset{\mathbb{K}}{\otimes} V_2, \tilde{f})$ will be unique up to unique iso.

<u>Example</u>: $V \underset{\mathbb{K}}{\otimes} \mathbb{K} \cong V \qquad v \otimes 1 \longleftrightarrow v$. $\quad \varphi(v,1) = v \otimes 1 = v$

$\tilde{f}(v) = f(v,1)$

<u>Construction</u>: $V_1 \underset{\mathbb{K}}{\otimes} V_2$ is a vector space spanned by $B = \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$. quotiented by the subspace H spanned by:

① $(z_1 v_1 + z_1' v_1', v_2) - z_1 (v_1, v_2) - z_1' (v_1', v_2)$)

& ② $(v_1, z_2 v_2 + z_2' v_2') - z_2 (v_1, v_2) - z_2' (v_1, v_2')$

So in the quotient: $(z_1 v_1 + z_1' v_1') \otimes v_2 = z_1 (v_1 \otimes v_2) + z_1' (v_1' \otimes v_2)$.

$v_1 \otimes (z_2 v_2 + z_2' v_2') = z_2 (v_1 \otimes v_2) + z_2' (v_1 \otimes v_2'$

• Define $\varphi(v_1, v_2) = v_1 \otimes v_2$ & check it is bilinear

①' $\varphi(z_1 v_1 + z_1' v_1', v_2) \overset{?}{=} z_1 \varphi(v_1, v_2) + z_1' \varphi(v_1', v_2)$ this is ①!

②' $\varphi(v_1, z_2 v_2 + z_2' v_2') \overset{?}{=} z_2 \varphi(v_1, v_2) + z_2' \varphi(v_1, v_2')$ —— ②!

⚠ The map is bilinear because we defined $V_1 \otimes_{\mathbb{K}} V_2$ as a quotient of vector spaces.