Lecture 35 : Tensor Product, Hom-Tensor adjointness

Recall: Last time we define tensor products & duals of vector spaces & maps

Def The dual of V, denoted by V^*, is defined as: $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$

Theorem 1: If V is finite-dimensional, then $\dim V^* = \dim V$. $\left(\underset{\text{basis}}{B} \rightsquigarrow \underset{\text{dual basis}}{B^*}\right)$

Q: How to dualize a map?

Lemma: If $f: V \longrightarrow W$ is a linear map, then $f^*: W^* \longrightarrow V^*$ defined by $f^*(\xi) = \xi \circ f: V \longrightarrow W \longrightarrow \mathbb{K}$ $\forall \xi \in W^*$ is $\mathbb{K}$-linear

More precisely: $f^*(\xi)(v) = \xi(f(v))$ $\forall v \in V, \forall \xi \in W^*$

Prop (HW 12): If $\dim V = n$, $\dim W = m$ then $[f^*]_{(B_W)^*(B_V)^*} = [f]^T_{B_V B_W}$.

Theorem 2 $V \xhookrightarrow{\varphi} (V^*)^*$ linear ; surj $\Leftrightarrow \dim V < \infty$

$v \longmapsto \varphi(v) : (f \longmapsto f(v))$ $\forall f \in V^*$

§.1 Bilinear Maps and Tensor Products:

Let V_1, V_2, W be 3 vector spaces over $\mathbb{K}$

Def: A bilinear map $f: V_1 \times V_2 \longrightarrow W$ is a set map which is linear in each coordinate, ie:

- $\forall v_1 \in V_1$: $V_2 \longrightarrow W$ $\in \mathrm{Hom}_{\mathbb{K}}(V_2, W)$

  $w \longmapsto f(v_1, w)$

- $\forall v_2 \in V_2$: $V \longrightarrow W$ $\in \mathrm{Hom}_{\mathbb{K}}(V_1, W)$

  $w \longmapsto f(w, v_2)$

Def: The Tensor product $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space together with a bilinear map $V_1 \times V_2 \xrightarrow{\varphi} V_1 \otimes_{\mathbb{K}} V_2$

$(v_1, v_2) \longmapsto v_1 \otimes v_2$ ← indecomposable tensor satisfying the following <u>universal property</u>: For any vector sp. W over $\mathbb{K}$ and any bilinear map $f: V_1 \times V_2 \longrightarrow W$, there exists a unique $\tilde{f}: V_1 \otimes V_2 \longrightarrow W$ linear map making the following diagram commute:

(*)
$$\begin{tikzcd}
V_1 \times V_2 \arrow[r, "f"] \arrow[d, "\varphi"'] & W \\
V_1 \otimes_{\mathbb{K}} V_2 \arrow[ur, dashed, "\exists ! \tilde{f} \text{ linear}"'] &
\end{tikzcd}$$

Obs: The vector space $V_1 \otimes_{\mathbb{K}} V_2$ is unique up to unique isomorphism
(HW12)

- Define: $V_1 \otimes_{\mathbb{K}} V_2 = \dfrac{\bigoplus_{v_1 \in V_1, v_2 \in V_2} \mathbb{K}(v_1, v_2)}{H}$

H = v.sp generated by:

① $(z_1 v_1 + z_1' v_1', v_2) - z_1 (v_1, v_2) - z_1' (v_1', v_2)$

② $(v_1, z_2 v_2 + z_2' v_2') - z_2 (v_1, v_2) - z_2' (v_1, v_2)$

Notation $\overline{(v_1, v_2)} = v_1 \otimes v_2$ in $V_1 \otimes V_2$

$\varphi: V_1 \times V_2 \xrightarrow{\pi} V_1 \otimes V_2$ natural projection

$(v_1, v_2) \longrightarrow \overline{(v_1, v_2)} = v_1 \otimes v_2$

φ is bilinear by ① & ②.

- **Q**: How to define $\tilde{f}$?

Use $\tilde{f}(v_1 \otimes v_2) = \tilde{f}(\varphi(v_1, v_2)) = f(v_1, v_2)$

- $\{v_1 \otimes v_2\}$ span, so it will be unique by construction.

We can use the universal property of $\oplus$ to get a ! linear map $g : \bigoplus_{\substack{v_1 \in V_1 \\ v_2 \in V_2}} \mathbb{K}(v_1, v_2) \longrightarrow W$ with $g(v_1,v_2) = f(v_1,v_2)$

+ extend linearly

($\{(v_1,v_2)\}$ is a basis for)

Now we want to check $g|_H = 0$.

(1'') $g((z_1 v_1 + z_1' v_1', v_2)) = f((z_1 v_1 + z_1' v_1', v_2))$

$= z_1 f(v_1,v_2) + z_1' f(v_1', v_2) = z_1 g(v_1,v_2) + z_1' g(v_1', v_2)$

f bilinear

So $g((z_1 v_1 + z_1' v_2', v_2) - z_1(v_1,v_2) - z_1'(v_1', v_2)) = 0$

(2'') Similarly $g((v_1, z_2 v_2 + z_2' v_2) - z_2(v_1,v_2) - z_2'(v_1, v_2')) = 0$

Conclusion: $H \subseteq \ker g$ so g gives a unique linear map

$$\tilde{f} : V_1 \otimes V_2 = \frac{\bigoplus_{v_i \in V_i} \mathbb{K}(v_1,v_2)}{H} \longrightarrow W.$$

- The diagram (*) commutes by construction. $\square$

Proposition: If $f_1 : V_1 \longrightarrow W_1$ & $f_2 : V_2 \longrightarrow W_2$ are $\mathbb{K}$-linear, then

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "(f_1, f_2)"] \arrow[dr, dashed, "\text{bilinear}"'] & W_1 \times W_2 \arrow[d, "\text{bilinear}"] \\ & W_1 \otimes W_2 \end{tikzcd}$$

$\rightsquigarrow$ get ! $f_1 \otimes f_2 : V_1 \otimes V_2 \longrightarrow V_1 \otimes W_2$ linear.

Lemma: $\dim_{\mathbb{K}}(V_1 \otimes V_2) = \dim_{\mathbb{K}} V_1 \cdot \dim_{\mathbb{K}} V_2$ (product of cardinalities)

Pf/ We write down a basis for $V_1 \otimes V_2$ as a product of bases.

If $B_{V_1} = \{ v_i^{(1)} : i \in I_1 \}$ is a basis for V_1 &

$B_{V_2} = \{ v_i^{(2)} : i \in I_2 \}$ ―――――― V_2, then

<u>Claim</u> $B = \{ v_i^{(1)} \otimes v_j^{(2)} \quad \substack{i \in I_1 \\ j \in I_2} \}$ is a basis for $V_1 \otimes V_2$

- <u>B spans $V_1 \otimes V_2$</u>:

It's enough to write the spanning set of indecomposable tensors as (finite) linear combinations of elements in B.

If $v_1 \in V_1$, $v_2 \in V_2$ then:

$v_1 = \sum_{i \in I_1} a_i \, v_i^{(1)}$ & $v_2 = \sum_{j \in I_2} b_j \, v_j^{(2)}$ with $a_i = 0$, $b_j = 0$

for all but finitely many $i \in I_1$, $j \in I_2$.

$$\text{Then } v_1 \otimes v_2 = \varphi\Big(\sum_{\substack{i \in I_1 \\ \text{finite}}} a_i v_i^{(1)}, \sum_{\substack{j \in I_2 \\ \text{finite}}} b_j v_j^{(2)} \Big)$$

$$\underset{\text{Reln ①}}{=} \sum_{\substack{i \in I_1 \\ \text{finite}}} a_i \varphi\Big(v_i^{(1)}, \sum_{j \in I_2} b_j v_j^{(2)} \Big)$$

$$\underset{\text{Reln ②}}{=} \sum_{\substack{i \in I_1 \\ j \in I_2 \\ \text{finite}}} \underbrace{a_i b_j}_{\in \mathbb{K}} \underbrace{\varphi(v_i^{(1)}, v_j^{(2)})}_{= v_i^{(1)} \otimes v_j^{(2)}}$$

$a_i b_j = 0$ for all but finitely many $i \in I_1$, $j \in I_2$.

- <u>B is l.i.</u>:

$$\sum_{\substack{i \in I_1 \\ j \in I_2 \\ \text{finite}}} c_{ij} \, v_i^{(1)} \otimes v_j^{(2)} = 0 \in V_1 \otimes V_2$$

<u>Want to show</u> $c_{ij} = 0$

Use def to rewrite it as
$$\sum_{\substack{j \in I_2 \\ \text{finite}}} \underbrace{\Big(\sum_{\substack{i \in I_1 \\ \text{finite}}} c_{ij} \, v_i^{(1)} \Big)}_{\in V_1} \otimes \underbrace{v_j^{(2)}}_{\in V_2} = 0.$$

($\mathbb{K}$ ☆)

Pick $\ell \in I_2$. We'll show $c_{i\ell} = 0 \ \forall i$

Since B_{V_2} is a basis for V_2: $\exists\ \psi : V_2 \longrightarrow \mathbb{K}$ linear with

$$\psi(v^{(2)}_j) = \delta_{j,\ell} \qquad \psi = (v^{(2)}_\ell)^* \in V_2^*.$$

By Proposition $\exists\ id_{V_1} \otimes \psi : V_1 \otimes V_2 \to \underbrace{V_1 \otimes_{\mathbb{K}} \mathbb{K}}_{= V_1}$ linear with

$$(id_{V_1} \otimes \psi)(v_1 \otimes v_2) = v_1 \otimes \psi(v_2).$$

Apply $id_{V_1} \otimes \psi$ to (**). Then:

$$\underset{\substack{\cap \\ V_1 \otimes_{\mathbb{K}} \mathbb{K} = V_1}}{0} = (id_{V_1} \otimes \psi)(0) = \sum_{\substack{j \in I_2 \\ \text{finite}}} \underbrace{\Big(\sum_{\substack{i \in I_1 \\ \text{finite}}} c_{ij}\, v_i^{(1)} \Big)}_{\in V_1} \otimes \underbrace{\psi(v_j^{(2)})}_{\in \mathbb{K}}$$

$\vec{v} \otimes a \Leftrightarrow a\vec{v}$

only $j=\ell$ survives

$$= \sum_{i \in I_1} c_{i\ell}\, v_i^{(1)} \otimes 1$$

$$\Rightarrow \underset{\substack{\uparrow \\ V_1}}{0} = \sum_{\substack{i \in I_1 \\ \text{finite}}} c_{i\ell}\, v_i^{(1)} \underset{B_{V_1} \text{ basis}}{\Longrightarrow} c_{i\ell} = 0 \quad \forall i$$

$\square$

Remark: Assume $\dim V_i = n_i < \infty$ & $\dim W_i = m_i < \infty$.

Assume $f_1 : V_1 \longrightarrow W_1$ linear are identified with matrices
$f_2 : V_2 \longrightarrow W_2$ —

$X_1 \in \mathrm{Mat}_{m_1 \times n_1}(\mathbb{K})$ & $X_2 \in \mathrm{Mat}_{m_2 \times n_2}(\mathbb{K})$. Then: $f_1 \otimes f_2$ gets identified with a matrix $X_1 \otimes X_2 \in \mathrm{Mat}_{(m_1 \cdot m_2) \times (n_1 \cdot n_2)}(\mathbb{K})$.

More precisely, if $X_1 = \begin{bmatrix} a_{11} & \cdots & a_{1n_1} \\ \vdots & & \\ a_{m_1 1} & \cdots & a_{m_1 n_1} \end{bmatrix}$ & $X_2 = \begin{bmatrix} b_{11} & \cdots & b_{1n_2} \\ \vdots & & \\ b_{m_2 1} & \cdots & b_{m_2 n_2} \end{bmatrix}$, then $X_1 \otimes X_2 = \begin{bmatrix} a_{11} X_2 & \cdots & a_{1n_1} X_2 \\ \vdots & & \\ \boxed{a_{m_1 1} X_2} & \cdots & a_{m_1 n_1} X_2 \end{bmatrix}$

$m_2 \times n_2$ matrix

$\mathcal{B}_{V_1} = \{ v_i^{(1)} \quad i=1,\dots,n_1 \}$
$\mathcal{B}_{V_2} = \{ v_i^{(2)} \quad i=1,\dots,n_2 \}$
$\mathcal{B}_{W_1} = \{ w_i^{(1)} \quad i=1,\dots,m_1 \}$
$\mathcal{B}_{W_2} = \{ w_i^{(2)} \quad i=1,\dots,m_2 \}$

Here we use $\mathcal{B}_{V_1 \otimes V_2} = \mathcal{B}_{V_1} \times \mathcal{B}_{V_2} = \bigcup_{i=1}^{n_1} \{ \{v_i^{(1)}\} \times \{v_j^{(2)}\} : j = 1,\dots,n_2 \}$

$\mathcal{B}_{W_1 \otimes W_2} = \mathcal{B}_{W_1} \times \mathcal{B}_{W_2} = \bigcup_{i=1}^{m_1} \{ \{w_i^{(1)}\} \times \{w_j^{(2)}\} : j = 1,\dots,m_2 \}$

(as in the Lemma)

§2. Hom-Tensor adjointness:

Prop: There is a natural map $V^* \otimes W \xrightarrow{\Phi} \mathrm{Hom}(V,W)$

with $\Phi(\xi \otimes w) : v \longmapsto \underbrace{\xi(v)}_{\in \mathbb{K}} \cdot w$.

Moreover, Φ is an isomorphism if V is finite-dimensional

In general: Φ is always injective.

Proof Define $\varphi : V^* \times W \longrightarrow \mathrm{Hom}(V,W)$

$(\xi, w) \longmapsto \{ v \longmapsto \xi(v) w \}$

- Easy check: φ is bilinear. Hence, it yields a unique linear map

$$\phi : V^* \otimes W \longrightarrow \mathrm{Hom}(V,W)$$

with $\phi(\xi \otimes w) = \varphi(\xi, w)$.

- φ is injective: Let $\alpha \in V^* \otimes W$ be such that $\phi(\alpha) = 0$

Write $\alpha = \sum_{j=1}^{N} \xi_j \otimes w_j$ (absorb scalars into ξ_j)

We can assume $w_1, \dots, w_N$ are linearly independent. Otherwise, use the dependency relation to reduce the number of w's.

(Eg rewrite $w_N = \sum_{j=1}^{N-1} a_j w_j \rightsquigarrow \sum_{j=1}^{N-1} \xi_j \otimes w_j + \sum_{j=1}^{N-1} \xi_N \otimes a_j w_j$

$$= \sum_{j=1}^{N-1} \xi_j \otimes w_j + \sum_{j=1}^{N-1} (a_j \xi_N \otimes w_j)$$

$$= \sum_{j=1}^{N} \underbrace{(\xi_j + a_j \xi_N)}_{\text{new } \xi_j'} \otimes w_j \quad)$$

Then: $\forall v \in V \quad \sum_{j=1}^{N} \xi_j(v) \, w_j = \phi(\alpha)(v) = 0$

$$\underset{\{w_1, .. w_N\} \text{ li}}{\Longrightarrow} \quad \xi_j(v) = 0 \quad \forall j = 1 \cdots N$$

But $\xi_j(v) = 0 \quad \forall v \in V \implies \xi_j = 0$

Conclude: $\alpha = \sum_{j=1}^{N} 0 \otimes w_j = 0$.

• Claim: If $\dim V = n < \infty$, then ϕ is surjective

Pf/ Let $B_V = \{v_1, \dots, v_n\}$ be a basis for V. Given $f \in \mathrm{Hom}_{\mathbb{K}}(V, W)$

let $w_i = f(v_i) \quad i = 1, \dots n$.

Then $\alpha = \sum_{i=1}^{m} v_i^* \otimes w_i \in V^* \otimes W$ satisfies $\phi(\alpha) = f$.

because $\sum_{i=1}^{m} \phi(v_i^* \otimes w_i)(v_j) = \sum_{i=1}^{m} \underbrace{v_i^*(v_j)}_{\delta_{ij}} \cdot w_i = w_j$

So $\phi(\alpha)$ & f agree on B_V, so they are the same function. □

In particular, when $W = V$ & $\dim V < \infty$, we use this to write a canonical tensor in $V^* \otimes V$, that is basis independent (next time)