Lecture 36: Canonical tensors & tensor algebra

Last Time: Defined $V \otimes_{\mathbb{K}} W$ of 2 $\mathbb{K}$-vector spaces V & W via universal property

$$\begin{tikzcd}
V \times W \arrow[r, "f \text{ bilin}"] \arrow[d, "\varphi", "\text{bil}"'] & W \\
V \otimes_{\mathbb{K}} W \arrow[ur, dashed, "\exists!\, \tilde{f} \text{ linear}"'] &
\end{tikzcd}
\qquad (v,w) \mapsto (v \otimes w)$$

Obs.: We can define $\bigotimes_{i=1}^{n} V_i = V_1 \otimes_{\mathbb{K}} V_2 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_n$ by working with multilinear forms $V_1 \times \cdots \times V_n \longrightarrow W$ (linear on each factor, once values on other factors are fixed) & an analogous universal property

(Take $(V_1 \otimes \cdots \otimes V_{n-1}) \otimes V_n$ & induct (see HW12 Problem 15) for $n=3$)

Prop: $\left.\begin{array}{l} B_V = \{v_i : i \in I\} \text{ basis for } V \\ B_W = \{w_j : j \in J\} \text{ — } W \end{array}\right\} \Rightarrow \{v_i \otimes w_j : \begin{array}{l} i \in I \\ j \in J \end{array}\}$ basis for $V \otimes_{\mathbb{K}} W$

Prop (Hom-Tensor adjointness) There is a natural map

$$V^* \otimes W \xrightarrow{\Phi} \mathrm{Hom}(V, W) \quad \text{with} \quad \Phi(\xi \otimes w) : v \longmapsto \underbrace{\xi(v)}_{\in \mathbb{K}} \cdot w$$

Moreover, Φ is an isomorphism if V is finite-dimensional

In general: Φ is always injective.

HW12: Many more properties of $\otimes$, including the rank of an element u in $V \otimes W$ ($= \min\{r \mid u = \sum_{i=1}^{r} v_i \otimes w_i$ for some $v_i \in V$, $w_i \in W\}$

indec. tensors

§1. Canonical Tensor:

In particular, when $W = V$ & $\dim V < \infty$, we use this to write a canonical tensor in $V^* \otimes V$, that is basis independent

Corollary: Let V be finite dimensional $(\dim V = n)$. Choose a basis $\{v_i\}_{1\le i\le n}$ for V and let $\{v_i^*\}_{1\le i\le n}$ be its dual basis. Then:

$\Omega := \sum_{i=1}^{n} v_i^* \otimes v_i$ is independent of the choice of basis.

Proof: If $\{\tilde{v}_i\}_{1\le i\le n}$ is another basis for V, then

$$\left.\begin{array}{l} \alpha = \sum_{i=1}^{n} v_i^* \otimes v_i \\ \beta = \sum_{i=1}^{n} \tilde{v}_i^* \otimes \tilde{v}_i \end{array}\right\} \in V^* \otimes V \quad \& \quad \Phi(\alpha) = \Phi(\beta) = \mathrm{Id}_V \in \mathrm{Hom}(V,V)$$

$$v = \sum_{i=1}^{n} a_i v_i = \sum_{j=1}^{n} b_j \tilde{v}_j$$

$$\Rightarrow \Phi(\alpha)(v) = \sum_{i=1}^{n} \Phi(v_i^* \otimes v_i)\Big(\sum_{j=1}^{n} a_j v_j\Big)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_j \Phi(v_i^* \otimes v_i)(v_j)$$

$$= \sum_{i,j=1}^{n} a_j \underbrace{v_i^*(v_j)}_{\delta_{ij}} \cdot v_i = \sum_{i=1}^{n} a_i v_i = v$$

Same reasoning gives $\Phi(\beta)(v) = v$.

Since Φ is an iso, $\alpha = \beta$. $\square$

Obs 1. If $\dim V$ is infinite, we can't use this to define a canonical tensor. We need to put a topology on $V \otimes V^*$ & take completion.

Obs 2: This also shows Φ in Hom-tensor adjointness cannot be surjective if V is infinite-dimensional.

Take $W=V$, then $\mathrm{id}_V \notin \mathrm{Im}\,\Phi$ (it would correspond to the canonical tensor, but we don't have one... (we can't have infinite sums in $V^*\otimes V$.))

§2 Tensor Algebra

Pick any $\mathbb{K}$-vector space V. Inductively we define

$$T^k(V) = V^{\otimes k} = \underbrace{V\otimes \cdots \otimes V}_{k \text{ terms}} \quad \text{for all } k\geq 2 .$$

$\rightsquigarrow$ Collect all these tensors into 1 vector space via $\oplus$.

$$T^\bullet(V) = \bigoplus_{n\geq 0} T^n(V) \qquad (T^0(V) = \mathbb{K})$$

<u>Prop</u>: $T^\bullet(V)$ is a $\mathbb{K}$-algebra $=$ $\mathbb{K}$-vector space + ring structure

<u>Proof</u>: Enough to define the multiplication between $T^n(V)$ & $T^m(V)$

$$T^n(V)\times T^m(V) \longrightarrow T^{n+m}(V) \quad \underline{\text{bilinear}}$$

. We define it on indecomposable Tensors & extend $\mathbb{K}$-bilinearly.

$$(v_1\otimes\cdots\otimes v_n)\cdot(w_1\otimes\cdots\otimes w_m) = v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\cdots\otimes w_m$$

. Well-defined? (1) We can work with bases & extend $\mathbb{K}$-linearly

(2) Universal Property:

$$\begin{tikzcd}
(\overbrace{V\times\cdots\times V}^{n \text{ copies}})\times(\overbrace{V\times\cdots\times V}^{m \text{ copies}}) \arrow[r, "\text{mult}"] \arrow[d, "(\varphi_n,\varphi_m)"] & T^{n+m}(V) \\
T^n(V)\times T^m(V) \arrow[ur, dashed, "\varphi"] \arrow[d, "\varphi"] & \\
T^n(V)\otimes T^m(V) \arrow[uur, "="'] &
\end{tikzcd}$$

$\Rightarrow T^n(V)\times T^m(V) \longrightarrow T^{n+m}(V)$

mult $=\varphi$ bilinear in def of

$T^n(V)\otimes T^m(V) = T^{n+m}(V)$

Example: $V = \mathbb{K}^n$, then $T^m(V) \leftrightarrow$ homogeneous degree m polynomials in n non-commuting variables.

- variables $x_i \longleftrightarrow e_i$ (standard basis for $\mathbb{K}^n$)

Eg. $(\sum_j a_i e_i) \otimes (\sum_j b_j e_j) = \sum_{i,j} a_i b_j (e_i \otimes e_j)$

$$e_i \otimes e_j \updownarrow x_i \cdot x_j \text{ monomial}$$

$\rightsquigarrow T^2(\mathbb{K}^n) =$ homog degree 2 polys in n non-commuting variables.

§3 Symmetric & Exterior Algebras

Next, we define two quotients of $T^k(V)$ that allow us to swap entries on tensors, with/without a sign. We assume char $\mathbb{K} \neq 2$

Definition:

- $S^k(V) = \mathrm{Sym}^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1 \otimes \cdots \otimes v_k - v_1 \otimes \cdots \otimes \underset{i}{v_{i+1}} \otimes \underset{i+1}{v_i} \otimes \cdots \otimes v_k \\ 1 \leq i \leq n-1 \\ \text{for } v_1, \ldots, v_k \in V \end{array} \right\rangle$

  (k^{th} symmetric product of V)

- $\Lambda^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1 \otimes \cdots \otimes v_k + v_1 \otimes \cdots \otimes \underset{i}{v_{i+1}} \otimes \underset{i+1}{v_i} \otimes \cdots \otimes v_k \\ 1 \leq i \leq n-1 \\ \text{for } v_1, \ldots, v_k \in V \end{array} \right\rangle$

  (k^{th} exterior (or alternating) product of V)

Obs 1: A typical summand of an element in $S^k(V)$ is written as $v_1 \cdot \cdots \cdot v_k$

- $S^k(\mathbb{K}^n) =$ deg k homogeneous polynomials in n commuting variables!

Eg: $S^2(\mathbb{K}^n) \ni \left(\sum_{i=1}^n a_i e_i\right) \otimes \left(\sum_{j=1}^n b_j e_j\right) = \sum_{i,j=1}^n a_i b_j \, e_i \otimes e_j$

$$= \sum_{i<j} (a_i b_j + a_j b_i)\, e_i e_j + \sum_{i=1}^n a_i b_i \, e_i e_i$$

$e_i \otimes e_j = e_j \otimes e_i$ in $S^2(\mathbb{K}^n)$

$$\Longleftrightarrow \sum_{i<j} (a_i b_j + a_j b_i)\, x_i x_j + \sum_{i=1}^n a_i b_i \, x_i^2 .$$

Lemma: $v_1 \otimes \cdots \otimes v_k - sg(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} = 0$ in $\Lambda^k V$ for all $\sigma \in S_k$, $v_1, \ldots, v_n \in V$

Pf/ By induction on $\mathrm{len}(\sigma)$ (write σ as a product of simple transpositions)

Base case $\mathrm{len}(\sigma) = 1$: is the definition of $\Lambda^k(V)$. $\square$

Obs 2: A typical summand of an element in $\Lambda^k(V)$ is written as $v_1 \wedge \cdots \wedge v_k$ (order matters!)

- $v_1 \wedge \cdots \wedge v_k = 0$ if $v_i = v_j$ for some $i \neq j$ (by Lemma)

  $\mathrm{sign}\,(ij) = -1$, so flipping signs introduces a negative sign

- $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \mathrm{sign}(\sigma)\, (v_1 \wedge \cdots \wedge v_k) \quad \forall \sigma \in S_k$.

Prop: Basis for $S^k(V)$ & $\Lambda^k(V)$:

If $\{v_i : i \in I\}$ is a basis for V & I is totally ordered, then

(1) $\left\{ v_{i_1}^{r_{i_1}} \cdots v_{i_s}^{r_{i_s}} : \begin{array}{l} r_{i_1} + \cdots + r_{i_s} = k \\ s \geq 1, \; r_{i_j} \geq 1 \end{array} \right\}$ is a basis for $S^k(V)$

(2) $\{ v_{i_1} \wedge \cdots \wedge v_{i_k} \;\; i_1 < i_2 < \cdots < i_k \text{ in } I \}$ is a basis for $\Lambda^k(V)$

Obs: Total order if I is infinite requires the axiom of choice.

Corollary: Assume $\dim V = n$. Then,

(1) $\dim S^k V = \binom{n+k-1}{k}$ (# monomials of deg k in n variables)

(2) $\dim (\Lambda^k V) = \binom{n}{k}$ ($=0$ if $k>n$)

We can define Symmetric and exterior algebras:

$$\mathrm{Sym}^\bullet(V) = S^\bullet(V) = \bigoplus_{n\geq 0} S^n(V) \qquad (S^0(V)=\mathbb{K}, S^1(V)=V)$$

$$\Lambda^\bullet(V) = \bigoplus_{n\geq 0} \Lambda^n(V) \qquad (\Lambda^0(V)=\mathbb{K}, \Lambda^1(V)=V)$$

Prop: $S^\bullet(V)$ & $\Lambda^\bullet(V)$ are $\mathbb{K}$-algebras.

Pf/ Follow the same idea as in $T^\bullet(V)$. Define multiplication

$$\Phi: S^n(V) \times S^m(V) \longrightarrow S^{n+m}(V)$$

$$\overline{v_1\otimes\cdots\otimes v_n} \times \overline{w_1\otimes\cdots\otimes w_m} \longmapsto \overline{v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\cdots\otimes w_m}$$

$$\Psi: \Lambda^n(V) \times \Lambda^m(V) \longrightarrow \Lambda^{n+m}(V)$$

$$v_1\wedge\cdots\wedge v_n \times w_1\wedge\cdots\wedge w_m \longmapsto v_1\wedge\cdots\wedge v_n\wedge w_1\wedge\cdots\wedge w_m$$

More precisely: $\Phi(\overline{u}, \overline{u'}) = \overline{\varphi(u,u')}$ in $S^{n+m}(V)$

$\Psi(\overline{u}, \overline{u'}) = \overline{\varphi(u,u')}$ in $\Lambda^{n+m}(V)$

To show it's well-defined, need to show the relations defining $S^k(V)$ & $\Lambda^k(V)$ are preserved (Exercise)

. multiplication is associative, distributive by construction $\square$