Lecture 37: More on Tensor Algebras, Minors of a matrix

Last Time: Defined $T^\bullet(V)$, $S^\bullet(V)$ & $\Lambda^\bullet(V)$

- $S^k(V) = \mathrm{Sym}^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l}\text{subspace spanned by} \\ v_1\otimes\cdots\otimes v_k - v_1\otimes\cdots\otimes v_{i+1}\otimes v_i\otimes\cdots\otimes v_k \quad 1\leq i\leq k-1 \\ \text{for } v_1,\ldots,v_k\in V\end{array}\right\rangle$

  (k^{th} symmetric product of V)

  $\boxed{k\geq 2}$

- $\Lambda^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l}\text{subspace spanned by} \\ v_1\otimes\cdots\otimes v_k + v_1\otimes\cdots\otimes v_{i+1}\otimes v_i\otimes\cdots\otimes v_k \quad 1\leq i\leq k-1 \\ \text{for } v_1,\ldots,v_k\in V\end{array}\right\rangle$

  (k^{th} exterior product of V)

$S^0(V) = \Lambda^0(V) = T^0(V) = \mathbb{K}$, $\quad S^1(V) = \Lambda^1(V) = T^1(V) = V$

$T^\bullet(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$, $\quad S^\bullet(V) = \bigoplus_{k=0}^{\infty} S^k(V)$, $\quad \Lambda^\bullet(V) = \bigoplus_{k=0}^{\infty}\Lambda^k(V)$

Bases: $\mathcal{B}\ \{v_i : i\in I\}$ basis for V $\qquad$ $\dim V = m$

For $V^{\otimes n} = \{v_{i_1}\otimes\cdots\otimes v_{i_n} \quad i_1,\ldots,i_n\in I\}$ $\qquad$ $\dim = m^n$

For $S^n(V) = \{v_{i_1}^{r_1}\cdots v_{i_s}^{r_s} \quad i_1<i_2<\cdots<i_s \quad r_1+\cdots+r_s = n,\ r_i\geq 0\}$ $\qquad$ $\dim = \binom{m+n-1}{n}$

For $\Lambda^n(V) = \{v_{i_1}\wedge\cdots\wedge v_{i_n} \quad i_1<i_2<\cdots<i_n\}$ $\qquad$ $\dim = \binom{m}{n}$

Prop: $T^\bullet(V)$, $S^\bullet(V)$, $\Lambda^\bullet(V)$ are $\mathbb{K}$-algebras.

mult: $V^{\otimes n}\times V^{\otimes m} \longrightarrow V^{\otimes(n+m)}$ $\quad$ in $T^\bullet(V)$

$(v_1\otimes\cdots\otimes v_n, w_1\otimes\cdots\otimes w_m) \mapsto v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\cdots\otimes w_n$

factors through $\quad S^n(V)\times S^m(V) \longrightarrow S^{n+m}(V)$ &

$\qquad\qquad\qquad \Lambda^n(V)\times\Lambda^m(V) \longrightarrow \Lambda^{n+m}(V)$.

§1 More on $T^\bullet(V)$, $S^\bullet(V)$, $\Lambda^\bullet(V)$:

Above description: $S^n(V)$ & $\Lambda^n(V)$ are quotients of $T^n(V)$.

Alternative approach: in char$(\mathbb{K})=0$ we can view $S^n(V)$ & $\Lambda^n(V)$ as subspaces of $T^n(V)$. & the multiplication respects the structure.

- Define an action of S_n on $T^n(V)$ via

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

(Need to show it extends from index tensors to $T^n(V)$, can do this via univ property $\sigma \bullet : \underbrace{V \times \cdots \times V}_{n \text{ times}} \longrightarrow T^n(V)$ multilinear $\leadsto \exists!\ \bar{\sigma} \cdot : T^n(V) \to T^n(V)$ with $\sigma(v_1, \ldots, v_n) = \bar{\sigma}(v_1 \otimes \cdots \otimes v_n)$)

- Define 2 operators $S, A : T^n V \longrightarrow T^n V$

$$S(\xi) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(\xi)$$

$$A(\xi) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, \sigma(\xi)$$

Prop: (1) $S^2 = S$, $A^2 = A$

(2) $\operatorname{Ker}(S) = \langle v_1 \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \rangle$ $\quad 1 \le i \le n-1 \quad v_1, \ldots, v_n \in V$

$\operatorname{Ker}(A) = \langle v_1 \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \rangle$ $\quad 1 \le i \le n-1 \quad v_1, \ldots, v_n \in V$.

So $\operatorname{Im}(S) \cong \dfrac{T^n(V)}{\operatorname{Ker}(S)} = \operatorname{Sym}^n(V)$

$\operatorname{Im}(A) \cong \dfrac{T^n(V)}{\operatorname{Ker}(A)} = \Lambda^n(V)$

Proof: See HW13.

signed action!

Obs: View $\operatorname{Sym}^n(V) = (T^n(V))^{S_n}$, $\Lambda^n(V) = (T^n(V))^{S_n, \varepsilon}$

Q: Compatibility with linear maps?

$$f: V \xrightarrow[\text{linear}]{} W \quad \rightsquigarrow \quad T^n(f): T^n(V) \longrightarrow T^n(W) \text{ linear}$$

$$S^n(f): S^n(V) \longrightarrow S^n(W) \text{ linear}$$

minors of matrices (Later today) $\rightarrow$ $\Lambda^n(f): \Lambda^n(V) \longrightarrow \Lambda^n(W)$ linear

- $T^n(f)(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n)$

(Via univ. prop.:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "f\times \cdots \times f", "\text{mult}"'] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & W \times \cdots \times W \arrow[r, "\varphi", "\text{mult}"'] & T^n(W) \\ T^n(V) \arrow[urr, dashed, "\exists!\, T^n(f)"', "\text{linear}"] & & \end{tikzcd}$$

)

- $S^n(f)(v_1 \cdots v_n) = f(v_1) \cdots f(v_n)$
- $\Lambda^n(f)(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$

Check $T^n(f)|_{\ker S} \subset \operatorname{Ker}(S)$, $\Lambda^n(f)|_{\operatorname{Ker}(A)} \subset \operatorname{Ker}(A)$

Q: Universal Properties?

Prop Given a $\mathbb{K}$ algebra A (unital, associative) & a $\mathbb{K}$-linear map $V \xrightarrow{\varphi} A$, then

- $\exists$ a unique extension: $\overline{\varphi}: T^\bullet(V) \longrightarrow A$. $(\overline{\varphi}|_V = \varphi)$
- If A is a commutative algebra, then $\exists!\, \overline{\varphi}: S^\bullet(V) \longrightarrow A$
- If A is skew-commutative, ie $ab = -ba$ $\forall a,b \in A$, then $\exists!\, \overline{\varphi}\ \Lambda^\bullet(V) \longrightarrow A$.

Prop: $\Phi\ V \otimes V \simeq S^2(V) \oplus \Lambda^2(V)$ $\quad$ ($\simeq$ even + odd func)

$$(v \otimes v') \longmapsto \left(\frac{v \otimes v' + v' \otimes v}{2} ; \frac{v \otimes v' - v' \otimes v}{2} \right)$$

Pf/ Well-defined via universal property:

Write $V \times V \longrightarrow S^2(V) \oplus \Lambda^2(V)$ bilinear

$(v, v') \longmapsto (v \cdot v', v \wedge v')$

$\Rightarrow$ This map factors through $V \otimes V$. This defines Φ.

We view $S^2(V)$ & $\Lambda^2(V)$ as subspaces of $\Lambda \otimes \Lambda$. & construct the inverse map Φ^{-1} via the inclusions $S^2(V) \hookrightarrow T^2(V)$

$\Lambda^2(V) \hookrightarrow T^2(V)$ $\square$

⚠ This decomposition does not extend beyond $n=2$. Instead

$$V^{\otimes n} \simeq \bigoplus_{\lambda \vdash n} \mathbb{S}^\lambda(V)$$

Schur functors

(partitions of n)

$\lambda_1 \geq \cdots \geq \cdots \geq 0$, $\sum_i \lambda_i = n$ $\lambda_i \in \mathbb{Z}_{\geq 0}$

Q: What happens to T^n, S^n & Λ^n when we consider direct sums?

<u>Lemma</u>: Consider 2 vector spaces V & W. Then $\forall n$:

(1) $$S^n(V \oplus W) = \bigoplus_{i=0}^{n} S^i(V) \otimes S^{n-i}(W)$$

(Think of polynomials in commuting variables x_i (for basis elements in V)

y_j ———————)

(2) $$\Lambda^n(V \oplus W) = \bigoplus_{i=0}^{n} \Lambda^i(V) \otimes \Lambda^{n-i}(W)$$

(3) $$T^n(V \oplus W) = \bigoplus_{k=0}^{n} \Big(\bigoplus_{i_1 + \cdots + i_k = n} \big(T^{i_1}(V) \otimes T^{i_2}(W) \otimes T^{i_3}(V) \otimes \cdots \big) \Big)$$

(Variables don't commute, so we can't rearrange putting V-part before W-piece)

Pf/ Pick bases for V & W & check both sides of each identity share the same natural bases.

§2 Determinants & minors.

Fix V, W finite dimensional $\mathbb{K}$-vector spaces & $f: V \longrightarrow W$ linear

Say $V \simeq \mathbb{K}^n$, $W \simeq \mathbb{K}^m$ (pick bases for V & W), so

$f \in \mathrm{Mat}_{m \times n}(\mathbb{K})$

$$\rightsquigarrow \quad \begin{array}{ccc} \Lambda^k(V) & \xrightarrow{\Lambda^k(f)} & \Lambda^k(W) \\ \wr\!\wr & \circlearrowleft & \wr\!\wr \\ \mathbb{K}^{\binom{n}{k}} & \dashrightarrow & \mathbb{K}^{\binom{m}{k}} \end{array}$$

$g \in \mathrm{Mat}_{\binom{m}{k} \times \binom{n}{k}}(\mathbb{K})$

basis $\{v_{j_1} \wedge \cdots \wedge v_{j_k}\}$
$\{w_{i_1} \wedge \cdots \wedge w_{i_k}\}$
$1 \leq j_1 < \cdots < j_k \leq m$
$1 \leq i_1 < \cdots < i_k \leq n$

Def: Minors of f are the entries of the matrix for g.

More precisely, write $\underline{i} = (i_1, \ldots, i_k) \in \binom{[m]}{k}$ (k-subsets of $[m]$)

$\underline{j} = (j_1, \ldots, j_k) \in \binom{[n]}{k}$ (——— $[n]$)

$[N] = \{1, \ldots, N\}$

$$\Lambda^k(f)_{\underline{i}, \underline{j}} = \det\left(f^{\underline{i}}_{\underline{j}}\right) = \boxed{\Delta^{\underline{i}}_{\underline{j}}(f)}$$

$\hookrightarrow$ submatrix of f with rows in $\underline{i}$ & columns in $\underline{j}$.

Q How do we compute $\Lambda^k(f)_{\underline{i}, \underline{j}}$?

Fix $B = \{v_1, \ldots, v_n\}$ a basis for V

$B' = \{w_1, \ldots, w_n\}$ ——— W

Write $f(v_j) = \sum_{i=1}^{n} a_{ij} w_i$

$$\Lambda^k(f)(v_{j_1} \wedge \cdots \wedge v_{j_k}) = \left(\sum_{i=1}^{m} a_{ij_1} w_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{m} a_{ij_k} w_i\right)$$

$\Rightarrow \Lambda^k(f)_{\underline{i}, \underline{j}} = $ coeff of $w_{i_1} \wedge \cdots \wedge w_{i_k}$ in $\Lambda^k(f)(v_{j_1} \wedge \cdots \wedge v_{j_k})$

Recall $w_{\sigma(i_1)} \wedge \cdots \wedge w_{\sigma(i_k)} = \mathrm{sgn}(\sigma)\, w_{i_1} \wedge \cdots \wedge w_{i_k} \quad \forall \sigma \in S_k.$

So Coeff of $w_{i_1} \wedge \cdots w_{i_k}$ in $\left(\sum_{i=1}^{m} a_{ij_1} w_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{m} a_{ij_k} w_i\right)$ (*)

is $\sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, a_{i_{\sigma(1)} j_1}\, a_{i_{\sigma(2)} j_2} \cdots a_{i_{\sigma(k)} j_k}$

In particular: $k = n = m$ write $\det(f) = \Delta^{1,\dots,n}_{1,\dots,n}(f)$

This recovers the permutation formula for determinants.

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}.$$

$$\underset{\tau = \sigma^{-1}}{=} \sum_{\tau \in S_n} \operatorname{sign}(\tau)\, a_{1\tau(1)}\, a_{2\tau(2)} \cdots a_{n\tau(n)}$$

Consequence 1 $\det(A) = \det(A^T)$ $\quad A \in \operatorname{Mat}_{n\times n}(\mathbb{K})$

Consequence 2 Row-expansion formula for $\det(A)$

Pf/ Fix i^{th} Row of A

$$\det(A) = \sum_{j=1}^{n} a_{ij} \underbrace{\sum_{\sigma:\, \sigma(j)=i} \operatorname{sign}(\sigma)\, a_{\sigma(1)1} \cdots \widehat{a_{\sigma(j),j}} \cdots a_{\sigma(n),n}}_{(**)}$$

restrict σ to $\tilde{\sigma} = \{1,\dots,\hat{j},\dots,n\} \xrightarrow[bij]{} \{1,\dots,\hat{i},\dots,n\}$ so

$$\operatorname{sign}(\sigma) = (-1)^{i+j} \operatorname{sign}(\tilde{\sigma}).$$

So $(**) = \det\left(A^{(i,j)}\right)(-1)^{i+j}$

$$\Rightarrow \det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A^{(i,j)}. \qquad \square$$

Obs: Using 1 & 2 we get column expansion formula as well.

Consequence 3 A with 2 equal rows (resp 2 equal cols), then $\det(A) = 0$.

Pf/ $\Lambda^n(f)(v_1\wedge\cdots\wedge v_n) = 0$ in $\Lambda^n(\mathbb{K}^n)\simeq\mathbb{K}$ by $(*)$ $\square$

Def (Cofactor matrix of A)

$$\mathrm{Cof}(A)_{i,j} = (-1)^{i+j}\det A^{(i,j)} \qquad 1\le i,j\le n.$$

Consequence 4 $\quad (\mathrm{Cof}\,A)^T A = \det(A)\, I_n = A(\mathrm{Cof}\,A)^T$

Pf/ $$((\mathrm{Cof}\,A)^T A)_{ij} = \sum_{l=1}^n (\mathrm{Cof}\,A)^T_{il}\, a_{lj}$$
$$= \sum_{l=1}^n (-1)^{i+l}\det(A^{(l,i)})\, a_{lj}$$

- If $i=j$ This is j^{th} column expansion of $\det(A)$.
- If $i\neq j$ ——————————— $\det(A')$ where

A' is the matrix obtained from A by replacing i^{th} col of A by the j^{th} col of A. By Consequence 3, $\det(A')=0$.

so $(\mathrm{Cof}\,A)^T A = \det A\, I_n$.

- $A(\mathrm{Cof}\,A)^T = ((\mathrm{Cof}\,A)A^T)^T = ((\mathrm{Cof}\,A^T)^T A^T)^T = (\det A^T\, I_n)^T$
$= \det A\, I_n$.

since
$$(\mathrm{Cof}\,A^T)_{ij} = (-1)^{i+j}\det(A^T)^{(i,j)} = (-1)^{i+j}\det A^{(j,i)} = (\mathrm{Cof}\,A)_{j,i}.$$
$\square$

Q. What happens if we do this for $S^k(V)\xrightarrow{S^k(f)} S^k(W)$

We get permanents!

Def: $\mathrm{Perm}(F)_{\underline{i},\underline{j}}$ = coeff of $w_{i_1}\cdots w_{i_k}$ in $S^k(f)_{(v_{j_1}\cdots v_{j_k})}$

$i_1<\cdots<i_k \quad j_1<\cdots<j_k$

⚠ We are NOT allowed to have repetitions, so we are not capturing all the coefficients of $S^k(f)$. Can include this by repeating columns.

In particular, for $n=m=k$, we have

$$\mathrm{Perm}(A) = \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

⚠ It is no longer true that matrices with repeated rows have permanent $=0$. This makes it very hard to compute! In particular, there are <u>NO</u> good algorithms for computing permanents.