Lecture 38: Determinants, Gaussian decompositions, bilinear forms

§.1 Minors:

Recall: $f: V \longrightarrow W$ $\dim V = n$, $\dim W = m$ $B = \{v_1, \dots v_n\}$
$B' = \{w_1, \dots w_m\}$

$(\underline{i}, \underline{j})$ minor: $\Delta^{\underline{i}}_{\underline{j}}(f) =$ coeff of $w_{i_1} \wedge \dots \wedge w_{i_k}$ in $\Lambda^k(f)(v_{j_1} \wedge \dots \wedge v_{j_k})$

$\underline{i} = \{i_1, \dots i_k\} \in \binom{[m]}{k}$ $\underline{j} = \{j_1, \dots, j_k\} \in \binom{[n]}{k}$

- For $n = m = k$

$$\begin{array}{ccc} \Lambda^n(f): \Lambda^n(V) & \longrightarrow & \Lambda^n(W) \\ \shortparallel & & \shortparallel \\ Sp(v_1 \wedge \dots \wedge v_n) & & Sp(w_1 \wedge \dots \wedge w_n) \\ \cong & & \cong \\ \mathbb{K} & \xrightarrow{\det(f)} & \mathbb{K} \end{array}$$

$\det(f) = \det([f]_{BB'})$ (HW 13) Problem 10

Prop: $\det(f \circ g) = \det(f) \cdot \det(g)$

Proof See HW 13 Problem 11.

§2. Gaussian Decomposition:

Fix $X \in GL_n(\mathbb{K})$

Def (HW 13 Pr 14): We say X admits a Gaussian Decomposition if

$$X = X^- X^0 X^+$$

where $X^0 =$ diagonal matrix

$$X^- = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$X^+ = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

Theorem 1: Gaussian decompositions are unique.

Pf/ $X^- X^0 X^+ = Y^- Y^0 Y^+$

$(Y^-)^{-1} X^- \; X^0 \; (X^+)(Y^+)^{-1} = Y^0$ Gaussian decomp of Y^0

since $$(Y^-)^{-1}X^- = \begin{pmatrix}1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix} = \begin{pmatrix}1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix}\begin{pmatrix}1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix} = \begin{pmatrix}1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix}$$

$$(X^+)(Y^+)^{-1} = \begin{pmatrix}1 & & 0 \\ & \ddots & \\ * & & 1\end{pmatrix}\begin{pmatrix}1 & & 0 \\ & \ddots & \\ * & & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & & 0 \\ & \ddots & \\ * & & 1\end{pmatrix}\begin{pmatrix}1 & & 0 \\ & \ddots & \\ * & & 1\end{pmatrix} = \begin{pmatrix}1 & & 0 \\ & \ddots & \\ * & & 1\end{pmatrix}$$

So it's enough to show it for diagonal matrices.

<u>Claim</u>: $X^- X^0 X^+ = Y^0$ diagonal $\Rightarrow X^- = X^+ = I_n$

Pf/ $\underbrace{X^0 X^+}_{\text{upper }\Delta} = \underbrace{(X^-)\, Y^0}_{\text{lower }\Delta}$ so both are diagonal (& invertible!)

$$X^0 = \begin{pmatrix}x_1 & & \\ & \ddots & \\ & & x_n\end{pmatrix} \Rightarrow X^0X^+ = \begin{pmatrix}x_1 & * & \cdots & * \\ & x_2 & & \vdots \\ & & \ddots & * \\ 0 & & & x_n\end{pmatrix} = \begin{pmatrix}y_1 & & & 0 \\ * & \ddots & & \\ \vdots & & \ddots & \\ * & \cdots & * & y_n\end{pmatrix}$$

$$Y^0 = \begin{pmatrix}y_1 & & \\ & \ddots & \\ & & y_n\end{pmatrix}$$

forces $x_1 = y_1,\ x_2 = y_2, \dots, x_n = y_n$

$$X^+ = \begin{pmatrix}1 & a_{12} & \cdots & a_{1n} \\ & 1 & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ & & & 1\end{pmatrix} \Rightarrow \underset{j>i}{0 =} (X^0X^+)_{ij} = \sum_{k=1}^n (X^0)_{ik}(X^+)_{kj}$$
$$= (X^0)_{ii}(X^+)_{ij}$$
$$= \underset{\substack{\neq \\ 0}}{x_i}\,(X^+)_{ij}$$

forces $(X^+)_{ij} = 0 \quad \forall\, j>i$ so $X^+ = I_n$.

Similarly: $X^- = I_n$ so $X^0 = Y^0$. $\square$

<u>Theorem 2</u>: X admits G.D $\iff \Delta^{1,\dots,i}_{1,\dots,i}(X) \neq 0\ \forall\, i = 1,\dots,n$

Pf/ $(\Rightarrow)$ $\Delta^{1,\dots,k}_{1,\dots,k}(X) = \Delta^{1,\dots,k}_{1,\dots,k}(X^- X^0 X^+)$

$$= \Delta^{1,\dots,k}_{1,\dots,k}\left(\left(\begin{array}{c|c}\begin{matrix}1 & & * \\ & \ddots & \\ 0 & & 1\end{matrix} & * \\ \hline 0 & \begin{matrix}1 & & * \\ & \ddots & \\ 0 & & 1\end{matrix}\end{array}\right)\left(\begin{array}{c|c}\begin{matrix}d_1 & & 0 \\ & \ddots & \\ 0 & & d_k\end{matrix} & 0 \\ \hline 0 & \begin{matrix}d_{k+1} & & 0 \\ & \ddots & \\ 0 & & d_n\end{matrix}\end{array}\right)\left(\begin{array}{c|c}\begin{matrix}1 & & 0 \\ & \ddots & \\ * & & 1\end{matrix} & 0 \\ \hline * & \begin{matrix}1 & & 0 \\ & \ddots & \\ * & & 1\end{matrix}\end{array}\right)\right)$$

$$= \det\begin{pmatrix}d_1 & & 0 \\ & \ddots & \\ 0 & & d_k\end{pmatrix} = d_1 \cdots d_k$$

So $\det(X) = d_1 \cdots d_n \neq 0 \iff$ all $d_i \neq 0$.

$\Rightarrow \Delta^{1,\dots,k}_{1,\dots,k}(\Delta) = d_1 \cdots d_k$ nonzero $\forall k$.

Moreover: $d_1 = \Delta^1_1(X)$

$d_1 d_2 = \Delta^{1,2}_{1,2}(\Delta) \rightsquigarrow d_2 = \dfrac{\Delta^{1,2}_{1,2}(X)}{\Delta^1_1(X)}$

In general $d_k = \dfrac{\Delta^{1,\dots,k}_{1,\dots,k}(X)}{\Delta^{1,\dots,k-1}_{1,\dots,k-1}(X)}$.

So X_0 is uniquely determined by the principal minors of X.

($\Leftarrow$) Explicitly, write

$$(X^0)_{kk} = \begin{cases} \Delta^1_1(X) & k=1 \\ \dfrac{\Delta^{1,\dots,k}_{1,\dots,k}(X)}{\Delta^{1,\dots,k-1}_{1,\dots,k-1}(X)} & k>1 \end{cases}$$

$$(X^-)_{ji} = \frac{\Delta^{1,\dots,i-1,j}_{1,\dots,i}(X)}{\Delta^{1,\dots,i}_{1,\dots,i}(X)}, \quad (X^+)_{ji} = \frac{\Delta^{1,\dots,i}_{1,\dots,i-1,j}(X)}{\Delta^{1,\dots,i}_{1,\dots,i}(X)}$$

and check $X = X^- X^0 X^+$ (by induction on n) $\square$

Obs: The construction is also true for $GL_n(R)$ where R is not necessarily a commutative ring (define dets via column expansion).

Obs 2: Non vanishing minors is an open condition in $\mathrm{Mat}_{n\times n}(\mathbb{K})$

We have a dense open set $\mathcal{U} \subseteq \mathrm{Mat}_{n\times n}(\mathbb{K})$ which we parameterize as

(Big Bruhat cell) $\mathcal{U} \cong \underset{(\text{entries in } X^-)}{\mathbb{K}^{\frac{n(n-1)}{2}}} \times \underset{\text{entries in } X^0}{(\mathbb{K}^*)^n} \times \underset{\text{entries in } X^+}{\mathbb{K}^{\frac{n(n-1)}{2}}}$

Can prove statements on $\mathrm{Mat}_{n\times n}(\mathbb{K})$ by restricting to $\mathcal{U}$.

§3. Bilinear forms:

Fix V_1, V_2, W $\mathbb{K}$-vector spaces

Recall: $\mathrm{Bil}_{\mathbb{K}}(V_1, V_2, W) = \{ f : V_1 \times V_2 \longrightarrow W \text{ bilinear} \}$

$f(v_1, -) \in \mathrm{Hom}_{\mathbb{K}}(V_2, W)$, $f(-, v_2) \in \mathrm{Hom}_{\mathbb{K}}(V_1, W)$

$\parallel$

$\mathrm{Hom}_{\mathbb{K}}(V_1 \otimes V_2, W)$

Def: A bilinear form on $V_1 \times V_2$ is an element of $\mathrm{Bil}(V_1, V_2, \mathbb{K})$

Def A bilinear form on $V_1 \times V_2$ is non-degenerate if

$$V_1 \hookrightarrow V_2^* \quad \& \quad V_2 \hookrightarrow V_1^*$$

$$v_1 \longmapsto f(v_1, -) \qquad v_2 \longmapsto f(-, v_2).$$

So if V_1 & V_2 have finite-dimensions, we get $V_1 \simeq V_2^*$, $V_2 \simeq V_1^*$ & $\boxed{\dim V_1 = \dim V_2}$

Motivation: Poincaré Duality

X smooth compact manifold of dim $= n$

(V1) $H_k(X) \otimes H^k(X) \xrightarrow{\int} \mathbb{R}$ is non-deg. $\forall\, 0 \le k \le n$

$$\sum_{i=1}^{m} a_i S_i \otimes \sum_{j=1}^{\ell} b_j \Psi_j \longmapsto \sum_{i,j} a_i b_j \int_{S_i} \Psi_i \qquad a_i, b_j \in \mathbb{R}$$

$S_i = k$-cell (simplicial) : $\Delta_k \hookrightarrow S_i \subseteq X$

Ψ_i : k-form on X (exterior) $\qquad \int_{S_i} \Psi_i = \int_{\Delta_k} S^* \Psi_i$ (Riemann integral)

(V2) $H^{n-k}(X) \otimes H^k(X) \longrightarrow \mathbb{R} \qquad \forall\, 0 \le k \le n$

$$\Psi \otimes \xi \longmapsto \int_X \Psi \wedge \xi$$

Poincaré duality: $H_k(X) \underset{(V1)}{\simeq} (H^k(X))^* \underset{(V2)}{\simeq} H^{n-k}(X) \qquad \forall\, 0 \le k \le n$

• Assume $\dim V_1 = \dim V_2 = n$ & pick $B_1 = \{v_1, \dots, v_n\}$ basis for V_1
$B_2 = \{w_1, \dots, w_n\}$ —— V_2

Write $f \in \mathrm{Bil}(V_1, V_2, \mathbb{K})$ via $\langle v_i, w_j \rangle = f(v_i, w_j)$

& build an $n \times n$ matrix $\boxed{Q}$ with $Q_{i,j} = \langle v_i, w_j \rangle \; \forall i, j$

Prop: f is non-degenerate if & only if Q is invertible.

Pf/ $(\Rightarrow)$ $V_1 \xrightarrow{\varphi_1} V_2^*$

$v \longmapsto \{w \mapsto f(v,w)\}$

So $\varphi_1(v_j)(w_i) = f(v_j, w_i) = Q_{ji}$

This means $\varphi_1(v_j) = \sum_{i=1}^{n} Q_{ji} w_i^*$

$[\varphi_1]_{B_1 B_2^*} = Q^T$

Since φ_1 is inj & $\dim V_1 = \dim V_2^* = n < \infty$, we conclude φ_1 is an iso. Thus, we conclude Q^T (and hence Q) is invertible.

Similarly $V_2 \xrightarrow{\varphi_2} V_1^*$

$w_j \longmapsto (v_i \mapsto f(v_i, w_j) = Q_{ij}$

So $\varphi_2(w_j) = \sum_{i=1}^{n} Q_{ij} v_i^*$ & so $[\varphi_2]_{B_2 B_1^*} = Q$.

Conclude Q is invertible.

$(\Leftarrow)$ Q is invertible & Q^T are invertible.

Also $[\varphi_1]_{B_1 B_2^*} = Q^T$, $[\varphi_2]_{B_2 B_1^*} = Q$, so

both φ_1 & φ_2 are invertible & f is non-degenerate. $\square$