Lecture 39: Bilinear forms II (Symmetric forms)

Last Time: $\mathrm{Bil}_{\mathbb{K}}(V_1, V_2, \mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(V_1 \otimes V_2, \mathbb{K}) = (V_1 \otimes V_2)^*$

- $f \in \mathrm{Bil}_{\mathbb{K}}(V_1, V_2, \mathbb{K})$ non deg $\Leftrightarrow$ induces injections

$$V_1 \xhookrightarrow{\varphi_1} V_2^* \qquad V_2 \xhookrightarrow{\varphi_2} V_1^*$$
$$v \longmapsto f(v,-) \qquad v \longmapsto f(-,v)$$

Prop: V_1, V_2 finite dimensional, f non-deg $\Longrightarrow \dim V_1 = \dim V_2$ ($\overset{?}{\Longleftarrow}$)

A: If $V_1 \simeq \mathbb{K}^n \simeq V_2$ write $B_1 = \{\vec{v}_1, \ldots, \vec{v}_n\}$ basis for V_1

$B_2 = \{\vec{w}_1, \ldots, \vec{w}_n\}$ ——— V_2

$f \in \mathrm{Bil}_{\mathbb{K}}(V_1, V_2, \mathbb{K}) \leadsto Q \in \mathrm{Mat}_{n\times n}(\mathbb{K})$

$$f(v,w) = [v]_{B_1}^T\, Q\, [w]_{B_2} \longleftrightarrow Q_{ij} = f(v_i, w_j)$$

Note: $[\varphi_1]_{B_1 B_2^*} = Q^T \qquad [\varphi_2]_{B_2 B_1^*} = Q$

Proposition: f is non-deg $\Longleftrightarrow$ Q is invertible

Q: How to relate matrices in $\mathrm{Bil}(\mathbb{K}^n, \mathbb{K}^n, \mathbb{K})$ & $\mathrm{End}(\mathbb{K}^n)$?

A: Write $V = \mathbb{K}^n$ (with standard basis)

$f \in \mathrm{Bil}(V, V, \mathbb{K}) \qquad f(v,w) = v^T Q w$ for some $Q \in \mathrm{Mat}_{n\times n}(\mathbb{K})$

$$\mathrm{Bil}(V, V, \mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(V \otimes V, \mathbb{K})$$
$$= (V \otimes V)^* \underset{\text{fin dim}}{=} V^* \otimes V^*$$
$$\underset{\substack{\text{pick iso} \\ V \simeq V^*}}{=} V \otimes V^* \underset{\text{Hom-}\otimes\text{ Adj}}{=} \mathrm{End}(V)$$

§1 Symmetric, skew-symmetric & alternating forms:

Def: We say f in $\mathrm{Bil}(V,V,\mathbb{K})$ is

1. symmetric if $f(v,w)=f(w,v)$ $\forall v,w\in V$
2. skew-symmetric if $f(v,w)=-f(w,v)$ $\forall v,w\in V$
3. alternating if $f(v,v)=0$ $\forall v\in V$.

same unless char $\mathbb{K}=2$

Ex 1 Dot Product on $V=\mathbb{R}^n$. is symmetric.

In general $f(x,y)=x^TQy$ with $Q=Q^T$ gives a symmetric bilinear form on $\mathbb{K}^n$.

Ex 2 $V=\mathbb{K}$ & $f(x,y)=xy$ is symmetric not alternating bilinear form. Skew symmetric when char $\mathbb{K}=2$

Ex 3 $f((x,y),(x',y'))=xy'-x'y=\det\begin{pmatrix}x & x'\\ y & y'\end{pmatrix}$ is skew-symmetric & alternating bilinear form on $\mathbb{R}^2$.

Ex 4 Pick $u\in\mathbb{R}^3$, $f_u(v,w)=u\cdot(v\times w)$ is alternating, skew-sym. form on $\mathbb{R}^3$.

Ex 5 $\dim V=n<\infty$ & $W=\mathrm{End}_{\mathbb{K}}(V)=\mathrm{Mat}_{n\times n}(\mathbb{K})$

$f\in\mathrm{Bil}(W,W,\mathbb{K})$ via $f(A,A')=\mathrm{Tr}(AA')$ Trace form

symmetric by construction.

Ex 6 $V=C[0,1]=\{h:[0,1]\to\mathbb{R}\text{ cont}\}$. inf-dim'l $\mathbb{R}$-v.s.

- $F: V\times V\longrightarrow\mathbb{R}$ via $F(g,h)=\int_0^1 g(x)h(x)\,dx$

F is a symmetric bilinear form.

- Pick $k:[0,1]^2\longrightarrow\mathbb{R}$ continuous:

$F_k: V\times V\longrightarrow\mathbb{R}$ via $F_k(g,h)=\iint_{[0,1]^2} g(x)h(y)k(x,y)\,dx\,dy$

- F_k bilinear but not symmetric. (unless k is).

Lemma: $f \in Bil(\mathbb{K}^n, \mathbb{K}^n, \mathbb{K})$ with associated matrix Q. Then

1. f is symmetric if & only if $Q = Q^T$ (symmetric matrix)
2. f is skew-symmetric if & only if $Q^T = -Q$ (skew-sym matrix)
3. f is alternating if & only if $Q^T = -Q$ & $Q_{ii} = 0 \ \forall i$.

Proposition: $Bil(V, V, \mathbb{K}) \overset{\varphi}{\cong} Bil^{sym}(V, V, \mathbb{K}) \oplus Bil^{skew\text{-}sym}(V, V, \mathbb{K})$

(for char $\mathbb{K} \neq 2$)

Pf/ $\varphi(f) = f_1 + f_2$

$$f_1(v,w) = \frac{f(v,w) + f(w,v)}{2}$$
$$f_2(v,w) = \frac{f(v,w) - f(w,v)}{2}$$

□

Alternative: $(V \otimes V)^* \simeq V^* \otimes V^* \simeq S^2(V^*) \oplus \Lambda^2(V^*)$

Lemma: If char $\mathbb{K} \neq 2$ $\dim V < \infty$, then $f \in Bil^{sym}(V, V, \mathbb{K})$ is completely determined by $f(v,v)$ $\forall v \in V$.

Pf/ Say we want to determine $f(v,w)$, then

$$\begin{aligned} f(v+w, v+w) &= f(v, v+w) + f(w, v+w) \\ &= f(v,v) + f(v,w) + f(w,v) + f(w,w) \\ &= f(v,v) + 2f(v,w) + f(w,w) \end{aligned}$$

So $$f(v,w) = \frac{f(v+w, v+w) - f(v,v) - f(w,w)}{2}$$

□

Q: How to work with degenerate symm. forms in $Bil(V, V, \mathbb{K})$?

A: Given $f \in Bil^{symm}(V, V, \mathbb{K})$, & $v \in V$, set

$$v^\perp = \{w \in V \mid f(v,w) = 0\}.$$

$Rad(f) = \{v \mid f(v, -) = 0 \in V^*\} \subseteq V$ subspace.

If $V' = V/Rad(f)$ View $V = \underset{\underset{V'}{\cong}}{\boxed{W}} \oplus Rad(f)$ (via a section)

Claim: $f|_{W\times W}$ is non-deg

Pf/ The matrix for f has the form
$$\begin{array}{ccc} & W & Rad(f) \\ & \left(\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array}\right) & \begin{array}{c} W \\ Rad(f)\end{array} \end{array}$$

by symmetry $\quad v \in Rad(f)$

If $w \in \ker(\varphi_1 : W \longrightarrow W^*)$
$$w' \longmapsto f(w,w')$$

then $f(w,w') = 0 \quad \forall w' \in W$.
But $f(w,v) = 0 \quad \forall v \in Rad(f)$ $\Big\}$ $f(w,v) = 0 \ \forall v \in V$ so $w \in Rad(f)$

Conclude $w \in W \cap Rad(f) = 0$, so $\varphi_1 : W \hookrightarrow W^*$.

Since f is symmetric $\varphi_2 : W \hookrightarrow W^*$ as well. $\square$

Obs: Same idea works for skew-symmetric forms (we have $v^\perp$)

§ 2 Sylvester's Theorem:

GOAL Classify symmetric non-deg bilinear forms on $V \cong \mathbb{R}^n$ via invariants

STEP 1: Degenerate vs non-degenerate

① 1^{st} invariant = rank of f $= rk([f]) =$ dim $V -$ dim $Rad(f)$

STEP 2: Classify non-deg symm forms = Sylvester's Thm

② 2^{nd} invariant = signature of f

Sylvester's Theorem: Fix $f : V\times V \longrightarrow \mathbb{R}$ non deg symm bil form

Then $\exists$ basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of V s.t. $f(\varepsilon_i, \varepsilon_j) = \pm \delta_{ij}$

Moreover, the # of >0 is independent of the bases.

Def Signature(f) = # 1's (For degenerate forms = (# 1's, # −1's))

Proof STEP 1 Gram-Schmidt algorithm:

INPUT: $\{v_1, \ldots, v_n\}$ basis for $V \simeq \mathbb{R}^n$

OUTPUT: $\{w_1, \ldots, w_n\}$ basis for $V \simeq \mathbb{R}^n$ with $f(w_i, w_j) = 0$ if $i \neq j$

$w_1 = v_1 \neq 0$, so $f(w_1, w_1) \neq 0$ ⚠ This need not happen!

$$w_2 = v_2 - \frac{f(w_1, v_2)}{f(w_1, w_1)} w_1 \quad \text{satisfies}$$

$$f(w_1, w_2) = f\Big(w_1, v_2 - \overbrace{\boxed{\frac{f(w_1, v_2)}{f(w_1, w_1)}}}^{\in \mathbb{R}} w_1\Big)$$

$$\underset{f(w_1,-)\ \text{linear}}{=} f(w_1, v_2) - \frac{f(w_1, v_2)}{f(w_1, w_2)} f(w_1, w_1) = 0$$

- $\{w_1, w_2\}$ is li since $\{v_1, v_2\}$ are. So $w_2 \neq 0$

For $k < n$: Assume we've constructed $\{w_1, \ldots, w_k\}$ satisfying

1. $Sp(v_1, \ldots, v_k) = Sp(w_1, \ldots w_k) \Rightarrow \{w_1, \ldots w_k\}$ li
2. $w_i \neq 0$ & $f(w_i, w_j) = 0 \quad \forall\, i \neq j \quad 1 \leq i, j \leq k$.

f non-deg, & $w_i \neq 0$ so $f(w_i, w_i) \neq 0 \ \forall i$. ⚠ This need not happen!

Then
$$w_{k+1} = v_{k+1} - \sum_{i=1}^{k} \frac{f(w_i, v_{k+1})}{f(w_i, w_i)} w_i$$

So
$$f(w_j, w_{k+1}) = f\Big(w_j, v_{k+1} - \sum_{i=1}^{k} \frac{f(w_i, v_{k+1})}{f(w_i, w_i)} w_i\Big)$$

$$= f(w_j, v_{k+1}) - \sum_{i=1}^{k} \frac{f(w_i, v_{k+1})}{f(w_i, w_i)} f(w_j, w_i)$$

$$\underset{\text{only } i=j \text{ survives}}{=} f(w_j, v_{k+1}) - \frac{f(w_j, v_{k+1})}{f(w_i, w_i)} f(w_i, w_i) = 0$$

By construction
1. $Sp(v_1, \ldots v_{k+1}) = Sp(w_1, \ldots w_{k+1})$, so
2. $\{w_1, \ldots w_{k+1}\}$ is li (& $w_i \neq 0 \ \forall i$)

STEP 2 : Set $s = \#\{i : f(w_i, w_i) > 0\}$.

Then, reorder $\{w_1, \dots, w_n\}$ so that

$$\begin{cases} f(w_i, w_i) > 0 & i = 1, \dots, s. \\ f(w_i, w_i) < 0 & i = s+1, \dots, n \end{cases}$$

$$\varepsilon_j = \begin{cases} \dfrac{w_j}{\sqrt{f(w_j, w_j)}} & \text{for } j = 1, \dots s \\ \dfrac{w_j}{\sqrt{-f(w_j, w_j)}} & \text{for } j = s+1, \dots, n \end{cases}$$

So $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis, $f(\varepsilon_i, \varepsilon_j) = 0$ for $i \neq j$ &

$$f(\varepsilon_j, \varepsilon_j) = \begin{cases} 1 & \text{for } j = 1, \dots s \\ -1 & \text{for } j = s+1, \dots, n \end{cases}$$

□

STEP 3 Show s is an invariant of f. (Next Time!)

⚠ There is an issue with STEP 1 of the proof. Even if f is non-deg symm, bilinear form, this does **not** mean that $f(v,v) \neq 0$ whenever $v \neq 0$. We will fix it next time!