Lecture 40: Bilinear forms III

Recall $f \in \mathrm{Bil}(V,V,\mathbb{K})$ symmetric if $f(v,w) = f(w,v)$ $\forall v,w$
skewsymm $f(v,w) = -f(w,v)$ $\forall v,w$.

GOAL: Classify (non-deg) symmetric / skew-symm forms.

§1 Building orthogonal bases for $\mathbb{K}^n$: ($\mathrm{char}\,\mathbb{K} \neq 2$)

⚠ Last time, we ~~build~~ studied symmetric non-deg bilinear forms on $\mathbb{R}^n$ & build an orthogonal basis $B = \{\varepsilon_1, \ldots \varepsilon_n\}$ for $\mathbb{R}^n$ with respect to $f \in \mathrm{Bil}^{\text{Symm}}_{\text{nondeg}}(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R})$., ie $f(\varepsilon_i, \varepsilon_j) = 0$ $\forall i, j$.
This was done using Gram-Schmidt method. At the center of it we had the assumption that $f(v,v) \neq 0$ whenever $v \neq 0$. But this is non-nec. true! Whenever this fails, the algorithm cannot proceed.

We need the following key lemma to bypass this issue. We assume $\mathbb{K}$ is any field with $\mathrm{char}\,\mathbb{K} \neq 2$. (this includes $\mathbb{R}$).

Lemma: Assume f is symmetric, non-deg $\mathbb{K}$-bilinear form, on $\mathbb{K}^n$ and $\mathrm{char}\,\mathbb{K} \neq 2$. Then:

1. $\exists\, v_0 \neq 0$ with $f(v_0, v_0) \neq 0$
2. $\mathbb{K}^n = \mathrm{Sp}(v_0) \oplus \langle v_0 \rangle^\perp$ where $\langle v \rangle^\perp = \{w \in \mathbb{K}^n : f(v,w) = 0\}$
3. $f|_{\langle v_0\rangle^\perp \times \langle v_0 \rangle^\perp}$ is a non-degenerate symmetric bilinear form.

Pf/ By a Lemma from Lecture 39, page 3, we know that f is completely determined by $\{f(v,v) : v \in V\}$. More precisely,

$$f(v,w) = \frac{f(v+w, v+w) - f(v,v) - f(w,w)}{2}$$

① If $f(x,x) = 0$ $\forall x \in \mathbb{K}^n$, then $f(v,w) = 0$ $\forall v,w$ & so $f \equiv 0$
This cannot happen since f is nondegenerate. Thus, $\exists\, v_0 \in \mathbb{K}^n$

with $f(v_0,v_0)\neq 0$.

② Build $W = \langle v_0\rangle^{\perp}$. By ① $v_0 \notin W$.

Since $W = \ker\left(\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{\varphi_1(v_0)} & \mathbb{K} \\ w & \longmapsto & f(v_0,w)\end{array}\right)$ we know it is a subspace of $\mathbb{K}^n$.

- Claim: $W \cap Sp(v_0) = \{0\}$

Indeed, $f(v_0,\lambda v_0) = \lambda \underbrace{f(v_0,v_0)}_{\neq 0}$ so $\lambda v_0 \in W \iff \lambda = 0$.

- By the Rank-Nullity Theorem, $\dim W + \underbrace{rk(\varphi_1(v_0))}_{=1 \text{ since } \varphi_1(v_0)\neq 0 \text{ in } (\mathbb{K}^n)^*} = n$

So $\dim W = n-1$.

- $\dim(W + Sp(v_0)) \underset{\hookrightarrow \text{Claim}}{=} n$ so $\mathbb{K}^n$ □

③ Need to show $\begin{array}{ccc} W & \overset{\tilde{\varphi}_1}{\hookrightarrow} & W^* \\ w & \longmapsto & (v \mapsto f(w,v))\end{array}$ is injective.

Pick $w \in \ker(\varphi_1)$ so $f(w,v) = 0 \quad \forall v \in W$

But $w \in \langle v_0\rangle^{\perp}$ so $f(w,\lambda v_0) = \lambda f(w,v_0) = 0$.

So $f(w,-) = 0 \in (\mathbb{K}^n)^*$ by ②.

This implies $w \in \ker(V \overset{\varphi_1}{\hookrightarrow} V^*) = \{0\}$.

By symmetry $W \overset{\tilde{\varphi}_2}{\hookrightarrow} W^*$ as well, so $\tilde{f} = f_1$

Proposition: Assume $\tilde{f}: V \times V \longrightarrow \mathbb{K}$ is non-deg bilinear form, $\dim V = m$. Then, there exists $B = \{\varepsilon_1,\ldots,\varepsilon_m\}$ basis for V with $f(\varepsilon_i,\varepsilon_j) = 0 \quad \forall i\neq j$. & $f(\varepsilon_i,\varepsilon_i)\neq 0$.

Pf/ By induction on $m = \dim V$. Write $V \simeq \mathbb{K}^m$.

$m=1$: nothing to check. Any $v \in \mathbb{K}\setminus\{0\}$ satisfies

$f(v,v) \neq 0$ since otherwise $f \equiv 0$.

Inductive step: By Lemma, $\exists\, v_0 \in V \setminus \{0\}$ & $W = \langle v_0 \rangle^{\perp}$ with $f(v_0, v_0) \neq 0$, $V = Sp(v_0) \oplus W$ & $f|_{W \times W}$ non-deg

$\dim W = m-1$, so by inductive hypothesis $\exists \{\varepsilon_2, \ldots, \varepsilon_{m-1}\}$ basis for W with $f(\varepsilon_i, \varepsilon_j) = 0$ $\forall i \neq j$ & $f(\varepsilon_i, \varepsilon_i) \neq 0$. Take $\varepsilon_1 = v_0$

$f(\varepsilon_1, \varepsilon_i) = 0$ $\forall i > 1$ (since $W \perp v_0$) & $f(\varepsilon_1, \varepsilon_1) \neq 0$. $\square$

§1 Symmetric forms over $\mathbb{R}$:

STEP 1: Reduce to the non-degenerate case. (last time)

Sylvester's Theorem: Fix $f: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ non deg symmetric bil form

Then $\exists$ basis $B = \{\varepsilon_1, \ldots, \varepsilon_n\}$ of $\mathbb{R}^n$ s.t. $f(\varepsilon_i, \varepsilon_j) = \pm \delta_{ij}$

Moreover, the # of >0 is independent of the bases. (= signature of f)

Proof: By Proposition: $\exists \{w_1, \ldots, w_n\}$ orthogonal bases for V with $f(w_i, w_i) \neq 0$

STEP 2 (last time) Normalize via $\varepsilon_i = \dfrac{w_i}{\sqrt{|f(w_i, w_i)|}}$ $\forall i$.

STEP 3: Show $= \#\{i\text{'s} : f(\varepsilon_i, \varepsilon_i) = 1\}$ is independent of B

Say we have 2 basis as in the Theorem:

$B = \{v_1, \ldots, v_p, v_{p+1}, \ldots, v_n\}$ with $f(v_i, v_i) = \begin{cases} 1 & i \leq p \\ -1 & i > p \end{cases}$

$B' = \{w_1, \ldots, w_q, w_{q+1}, \ldots, w_n\}$ — $f(w_i, w_i) = \begin{cases} 1 & i \leq q \\ -1 & i > q \end{cases}$

$f(v_i, v_j) = f(w_i, w_j) = 0$ $\forall i \neq j$

- Assume $p < q$ & reach a contradiction. Symmetry yields $p = q$

We define a linear transformation $L: \mathbb{R}^n \longrightarrow \mathbb{R}^{p+n-q}$

$$\xi \longmapsto \begin{bmatrix} f(v_1, \xi) \\ \vdots \\ f(v_p, \xi) \\ f(w_{q+1}, \xi) \\ \vdots \\ f(w_n, \xi) \end{bmatrix}$$

Since $p+n-q<n$, the rank-nullity theorem gives $\xi_0 \in \ker L \setminus \{0\}$.

<u>Claim</u>: $\xi_0 \in Sp(v_{p+1}, \dots v_n) \cap Sp(w_1, \dots, w_q)$

Indeed $\xi_0 = \sum_{i=1}^{n} a_i v_i \Rightarrow 0 = f(v_i, \xi_0) = a_i \underbrace{f(v_i,v_i)}_{\neq 0} \ \forall i \leq p$

so $\xi_0 \in Sp(v_{p+1} \dots v_n)$

$\xi_0 = \sum_{i=1}^{n} b_i w_i \Rightarrow 0 = f(w_i, \xi_0) = -b_i \underbrace{f(w_i,w_i)}_{\neq 0} \ \forall i > q$

Now $f(\xi_0, \xi_0) = f\left(\sum_{i=p+1}^{n} a_i v_i, \sum_{i=p+1}^{n} a_i v_i\right)$

$= \sum_{i,j=p+1}^{n} a_i a_j f(v_i, v_j) = \sum_{i=p+1} -a_i^2 < 0$ ($\xi_0 \neq 0$)

$f(\xi_0, \xi_0) = f\left(\sum_{i=1}^{q} b_i w_i, \sum_{i=1}^{q} b_i w_i\right)$

$= \sum_{i,j=1}^{q} b_i b_j f(w_i, w_j) = \sum_{i=1}^{q} b_i^2 > 0$ ($\xi_0 \neq 0$)

This is a contradiction! $\square$

<u>Consequence</u>: Classification of quadratic forms q in $\mathbb{R}^n$

($=$ homogeneous degree 2 polynomials in $\mathbb{R}[x_1 \dots x_n]$)

After a linear change of coordinates, they become

$x_1^2 + \dots + x_s^2 - x_{s+1}^2 - \dots - x_r^2$

Pf/ $f(v,w) = \frac{1}{2}\left(q(v_1+w_1, \dots, v_n+w_n) - q(\underline{v}) - q(\underline{w})\right)$

is a bilinear form $r = \text{rank}(f)$, $s = \text{signature}(f)$

Note: $q(\xi) = f(\xi, \xi)$

• Change of basis 1: $\{e_1, \dots, e_n\} \longrightarrow \{w_1, \dots, w_r, \underbrace{w_{r+1}, \dots, w_n}_{\text{in } Rad(f)}\}$

Next, we view $W = Sp(w_1, \dots w_r) = \mathbb{R}^r$ & Consider

$$\tilde{f} = f|_{W \times W} : \mathbb{R}^r \times \mathbb{R}^r \longrightarrow \mathbb{R} \quad \text{symm, non-deg bilinear form}$$

with s = signature of $(\tilde{f})$

By Sylvester's Theorem $\exists$ basis $\{\varepsilon_1, \dots, \varepsilon_r\}$ of $\mathbb{R}^r = W$ with

with $f(\varepsilon_i, \varepsilon_i) = \begin{cases} 1 & i \leq s \\ -1 & i > s \end{cases}$ & $f(\varepsilon_i, \varepsilon_j) = 0 \ \forall i \neq j$.

• Change of basis 2: in W: $\{w_1, \dots w_r\} \longrightarrow \{\varepsilon_1 \dots \varepsilon_r\}$

Conclusion: $\{e_1, \dots, e_n\} \xleftrightarrow{\varphi} \{\varepsilon_1, \dots \varepsilon_r, w_{r+1}, \dots w_n\}$ φ linear change of coordinates

$$q \longmapsto \tilde{q}(\underline{x}) = x_1^2 + \dots + x_s^2 - x_{s+1}^2 - \dots - x_r^2.$$

§2. Symmetric bilinear $\mathbb{K}$-forms (char $\mathbb{K} \neq 2$)

$V \simeq \mathbb{K}^n$ $\quad f \in Bil(V, V, \mathbb{K})$ symmetric we can pick a basis for V $\quad B = B_1 \cup B_2$ where $Sp(B_2) = Rad(f)$ &

$\tilde{f} = f|_{W \times W} : W \times W \longrightarrow \mathbb{K}$ is non-degenerate $(W = Sp(B_1))$

$$[f]_{BB} = \begin{array}{cc} & \begin{array}{cc} m & \end{array} \\ m & \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array}\right] \end{array} \qquad Q = Q^T \text{ in } GL_m(\mathbb{K}).$$

$$Rank(f) = rk\, Q$$

• Q is symmetric & $Q = [\tilde{f}]_{B_1 B_1}$ $\quad rk(f) = rk(\tilde{f}) = rk(Q)$

Proposition: Assume $\tilde{f} : \mathbb{K}^m \times \mathbb{K}^m \longrightarrow \mathbb{K}$ is non-deg bilinear form & char $\mathbb{K} \neq 2$. Then, there exists $B = \{\varepsilon_1, \dots, \varepsilon_m\}$ basis for $\mathbb{K}^m$ with

$f(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j$ & $f(\varepsilon_i, \varepsilon_i) \neq 0 \quad \forall i$.

If $\mathbb{K} = \overline{\mathbb{K}}$ we can further require $f(\varepsilon_i, \varepsilon_i) = 1 \quad \forall i = 1, \dots, m$.

Proof: The first part of the statement is the Proposition on page 2.
To finish normalize the basis $\{\varepsilon_1', \dots, \varepsilon_m'\}$ from the Proposition as follows: let $a_i := f(\varepsilon_i', \varepsilon_i') \neq 0$ in $\mathbb{K} = \overline{\mathbb{K}}$, so $\exists\, \alpha_i \in \mathbb{K} \setminus \{0\}$ with $\alpha_i^2 = a_i$. Then $\varepsilon_i = \dfrac{\varepsilon_i'}{\alpha_i}$ satisfies

$$f(\varepsilon_i, \varepsilon_i) = \frac{1}{\alpha_i^2} f(\varepsilon_i', \varepsilon_i') = \frac{a_i}{a_i} = 1 \qquad \forall i = 1, \dots, m$$

$$f(\varepsilon_i, \varepsilon_j) = \frac{1}{\alpha_i \alpha_j} f(\varepsilon_i', \varepsilon_j') = 0 \qquad \forall i \neq j$$

□

§3. Skew-symmetric bilinear $\mathbb{K}$-forms (char $\mathbb{K} \neq 2$)

As we mentioned last time, the calculations for turning a deg bil. symmetric form into a non-deg one works for skew-symm. forms as well. So we focus on non-deg skew-symm forms over $\mathbb{K}$ with char $\mathbb{K} \neq 2$. Fix

- In what follows we present the analog of Sylvester's Thm for skew-symm non-deg $\mathbb{K}$-bilinear forms on $V \simeq \mathbb{K}^n$. This result is used to build Darboux coordinates in real symplectic manifolds.
- We start discussing the dimension of V.

Lemma: If $\exists\, f \in \mathrm{Bil}^{\text{skew-sym}}_{\text{non-deg}}(V, V, \mathbb{K})$ & $\dim V < \infty$ then $\dim V$ is even.

Proof: Pick $Q = [f]_{\mathcal{B}\mathcal{B}} \in \mathrm{Mat}_{n \times n}(\mathbb{K})$ where $\mathcal{B}$ is any basis for V.
Then $Q^T = -Q$ since f is skew-symm. & Q is invertible because f is nondegenerate.
Then $\det(Q) = \det(-Q^T) = (-1)^n \det Q^T = (-1)^n \det Q$
Since $\det Q \neq 0$ & char $\mathbb{K} \neq 2$ we have $(-1)^n = 1$ so n is even. □

Theorem: Let $f: V \times V \longrightarrow \mathbb{K}$ be a non-deg skew-symmetric form on a finite-dimensional $\mathbb{K}$-vector space V, with char $\mathbb{K} \neq 2$.
Then $\exists$ basis $B = \{\varepsilon_i, \eta_i\}_{i=1}^{n}$ for V s.t

① $f(\varepsilon_i, \eta_j) = \delta_{ij} = -f(\eta_j, \varepsilon_i) \quad \forall i,j$

② $f(\varepsilon_i, \varepsilon_j) = 0 = f(\eta_i, \eta_j) \quad \forall i,j.$

$$[f]_{BB} = \begin{bmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$$

(Alt: $[f]_{\varepsilon, \eta} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$)

Proof: By Lemma $\dim V$ is even ($= 2n$ for some $n \in \mathbb{Z}_{\geq 1}$)
We proceed by induction on n.

Base case: $n=1$ Pick any $e_1 \in V \setminus \{0\}$ & choose $\eta_1' \in V$ st $f(\varepsilon_1, \eta_1') \neq 0$ (such η_1 exists since $f(\varepsilon_1, -) \neq 0$ in V^*)

Claim: $\{\varepsilon_1, \eta_1'\}$ is li.

Otherwise, $\eta_1' = \alpha \varepsilon_1$ (since $\varepsilon_1 \neq 0$), so $f(\varepsilon_1, \eta_1') = \alpha \underbrace{f(\varepsilon_1, \varepsilon_1)}_{=0} = 0$. Contr!

Next, we rescale η_1' to $\eta_1 = \dfrac{\eta_1'}{f(\varepsilon_1, \eta_1')}$ so $f(\varepsilon_1, \eta_1) = 1$.
. $\dim V = 2$ so we are done.

Inductive Step: Pick $\{\varepsilon_1, \eta_1\}$ as in the base case.
Next, consider $W = \{v \in V : f(\varepsilon_1, v) = f(\eta_1, v) = 0\}$
By the Lemma below, we have:

① $\dim W = 2n - 2 = 2(n-1)$

② $W \oplus Sp(\varepsilon_1, \eta_1) = V$

③ $f|_{W \times W}$ is non-deg & skew-sym.

So by the IH, W has a basis $\{\varepsilon_i, \eta_i\}_{i=2}^{n}$ satisfying the conditions in the statement. The conditions
$f(\varepsilon_1, \varepsilon_j) = f(\eta_1, \eta_j) = 0 \quad \forall j \neq 1$ follow from the def of W.
$f(\varepsilon_1, \eta_j) = f(\eta_1, \varepsilon_j) = 0 \quad \forall j \neq 1$ ———— $\square$

Lemma: Let f be a non-deg skew-symm form on $V \simeq \mathbb{K}^{2n}$ with char $\mathbb{K} \neq 2$. Assume V admits 2 vectors li ε_1, η_1 with $f(\varepsilon_1, \eta_1) = 1$. Consider $W = \langle \varepsilon_1, \eta_1 \rangle^{\perp} = \{ w \in V : f(\varepsilon_1, w) = f(\eta_1, w) = 0 \}$.

Then: $V = Sp(\varepsilon_1, \eta_1) \oplus W$ & $f|_{W\times W}$ is non-deg.

Proof: We start by showing the sum is direct.

Claim 1: $W \cap Sp(\varepsilon_1, \eta_1) = \{0\}$.

Pf/ Pick $\omega = \alpha \varepsilon_1 + \beta \eta_1 \in W$. We show $\alpha = \beta = 0$ as follows.

(1) $0 = f(\varepsilon_1, \alpha\varepsilon_1 + \beta\eta_1) = \alpha \underbrace{f(\varepsilon_1, \varepsilon_1)}_{=0} + \beta \underbrace{f(\varepsilon_1, \eta_1)}_{=1} = \beta \Rightarrow \beta = 0.$

(2) $0 = f(\eta_1, \alpha\varepsilon_1) = \alpha f(\eta_1, \varepsilon_1) = -\alpha \Rightarrow \alpha = 0.$

Claim 2: $\dim W = \dim V - 2$

Pf/ $\varphi: V \longrightarrow \mathbb{K}^2$ is linear & $W = \operatorname{Ker}\varphi$

$$v \longmapsto \begin{bmatrix} f(\varepsilon_1, v) \\ f(\eta_1, v) \end{bmatrix}$$

$\varphi(-\varepsilon_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\varphi(\eta_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\operatorname{Im}\varphi = \mathbb{K}^2$

By Rank-Nullity Theorem, $\dim W = \dim V - 2$.

Claim 3: $f|_{W\times W}$ is non-deg.

Pf/ $\varphi_1: W \longrightarrow W^*$, $v \longmapsto f(v, -)$ $\qquad$ $\varphi_2 = -\varphi_1: W \longrightarrow W^*$, $v \longmapsto f(-, v)$

Pick $v \in \ker(\varphi_1)$ so $f(v, w) = 0$ $\forall w \in W$.

Now $v \in W$ so $f(v, \varepsilon_1) = f(v, \eta_1) = 0$.

Conclude $f(v, -) = 0$ in V^* $\Longrightarrow$ $v = 0$ since f is non-deg. $\square$