# Lecture 1: Course outline & Introduction to groups

## §1 Overview:

$\boxed{\text{① Group Theory}}$ (5 weeks)

- Basic definitions (gp, subgroups, normal subgps, homomorphisms)
- Isomorphism Thms
- Presentations by gens & relations
- Group Actions + Counting Lemmas
- p-groups & Sylow Thms
- Simple groups (Example: $A_n$ for $n \geq 5$)
- Classification of finitely generated abelian gps
- Direct / Semidirect Products
- Automorphisms of cyclic gps
- Solvable & Nilpotent groups
- Composition series & Jordan-Hölder

NICE EXAMPLES

- $S_n$ perm on $[n] = \{1, \dots, n\}$
- $D_n$ (symm of n-gon)
- $GL_N(K)$ (K field)
- $Aut(X)$ X graph
- $Pic(\text{graph})$, $Jac(\text{curve})$
- Coxeter gps (Weyl gps)
- Fundamental gps $\pi_i(X)$
- Cohomology groups
- Group of isometries of a metric space $(X, d)$
- Valued gp (DVR)

(& many more ....)

# 2 Ring Theory

(6 weeks)

- Basic definitions (rings, subrings, left/right ideals, homs ..)
- Isomorphism Thms
- Modules over rings ( $\oplus$, $\Pi$, homs )
- Prime / Maximal ideals
- Prime avoidance, Chinese Remainder Thm.
- Rings & modules of fractions, $\boxed{\text{Localization}}$

  (essential for Alg. Geometry!)
- Nil- & Jacobson Radicals
- Artinian & $\boxed{\text{Noetherian rings (f.g. structures)}}$

Hilbert Basis Thm.

Primary Decomposition

(Topological & alg. structure of solutions to polynomial syst)

- PID & Modules over PID

  $\rightsquigarrow$ Classification Thm

  (Application: finite ab gps)
- Number Theory connections
  - UFD $\rightsquigarrow$ Gauss' Lemma & Eisenstein criterion
  - Euclidean domains / Dedekind domains
  - $\mathbb{Z}[\sqrt{-D}]$ (quadratic integers)
- Invariant Theory:
  - Symmetric polynomials
  - discriminants (A-disc)
- Extras:
  - Valuation rings (DVRs)

    $\rightsquigarrow$ non-Archimedean geometry
  - Gröbner basis.

    (Computational Alg Geom)

## 3 Linear & Multilinear Algebra (3 weeks)

- Vector spaces over $\mathbb{K}$
  - basis, $\oplus$, $\mathrm{Hom}_{\mathbb{K}}(V,W)$, duals, $\otimes_{\mathbb{K}}$
  - $\boxed{V^* \otimes W \simeq \mathrm{Hom}_{\mathbb{K}}(V,W)}$ (Hom-tensor adjointness)
- $B(\cdot,\cdot): V_1 \times V_2 \longrightarrow K$ bilinear forms
- $B : V_1 \times \cdots \times V_N \longrightarrow K$ multilinear forms

$\rightsquigarrow$ $\boxed{\text{Universal properties defining } V_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_n}$

- Non-degenerate bilinear forms $B: V \times V \longrightarrow K$

  $\rightsquigarrow$ $\boxed{\Omega_B \in V \otimes V^* \text{ canonical tensor}}$ (indep. of basis!)
- Matrix repr-n of symm. non-deg bilinear forms.
- Over $\mathbb{R}$: Can diagonalize this matrix $\rightsquigarrow$ (rank, signature)

  $\rightsquigarrow$ positive def / semidef matrices. [Sylvester's Thm]
- Tensor Algebra: $T(V) = \bigoplus_{n=0}^{\infty} T^n(V)$ $\quad [T^n(V) = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}]$

$\rightsquigarrow$ $\boxed{\mathrm{Sym}^\bullet(V) \quad \& \quad \Lambda^\bullet(V)}$ $\rightsquigarrow$ Decomposition Thm

(symmetric) (exterior)

Properties, interplay with $\oplus$ & $^*$

- Determinants & minors via skew-symmetric forms
- $\boxed{\text{Cayley-Hamilton Thm}}$

  $\rightsquigarrow$ Nakayama's Lemma

  $(A, m) \circlearrowright \underset{\text{f.g.}}{M}$ & $mM = M \Rightarrow M = 0$
- Symplectic & orthogonal gps of matrices
- Decomposition Thm for matrices
  - polar
  - Jordan
  - Gaussian

## §1 Basics on groups & some examples:

**Definition**: A group $G$ is a set together with:

(1) a function $G \times G \longrightarrow G$ (group operation or multiplication)

$$(a, b) \longmapsto a * b$$

just notation

(2) an element $e \in G$, called unit / identity / neutral element

satisfying the following 3 properties:

(i) Associativity: $(a*b)*c = a*(b*c)$ for all $a, b, c \in G$.

(ii) $e$ Neutral: $e*a = a*e = a \quad \forall a \in G$.

(iii) Existence of Inverses: for every $a \in G$, there exists $b \in G$ such that $a*b = e = b*a$

Notation: $b = a^{-1}$ if $*$ is "multiplication" ⇝ non-abelian notation

$b = -a$ if $*$ is "addition" ⇝ abelian setting

**Definition** $(G, *)$ is abelian or commutative if $a*b = b*a \quad \forall a, b \in G$

# EXAMPLES:

1. $G = \mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ (integers)

   $a * b = a + b \qquad e = 0$ (abelian!)

   Inverse of $a = -a$

2. $G = \mathbb{R}_{>0}$ (positive reals)

   $a * b = a \cdot b$ (usual multiplication), $e = 1$

   Inverse of $a = \frac{1}{a}$ (abelian!)

## Non-examples:

1. $G = \mathbb{R}_{>0}$ but $a * b = a^b$ $(= \exp(b \ln a))$, $e = 1$

   **Issue**: Not associative!

2. $G = \mathbb{R} \cup \{-\infty\}$ $\quad a * b = \max\{a, b\}$, $e = -\infty$ (tropical addition $= \oplus$)

   **Issue**: No inverses (except when $a = \infty$)

# Weaker structures: monoid & semigroups

**Def**: A monoid is a set $G$ with $*: G \times G \longrightarrow G$ & $e \in G$ where the operation $*$ satisfies (i) Associativity
(ii) Existence of Neutral Element ($e$)

**Examples**:

① All groups are monoids.

② $(\mathbb{R} \cup \{-\infty\}, \oplus)$ is a monoid.

③ $\mathbb{N} = \{0, 1, 2, 3 \dots\}$, $*$ = usual addition $e = 0$.

④ $E$ set $X = 2^E = \{\text{subsets of } E\} = \mathcal{P}(E)$ $A * B = A \cup B$
$e = \emptyset$.

**Def**: A semigroup is a set $G$ with $*: G \times G \longrightarrow G$ (law of composition) satisfying Associativity

**Example**: $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$ with $+$ is semigp, not monoid
$\cdot$ is monoid.

# More examples of groups

③ $G = GL_2(\mathbb{R})$ = real 2x2 matrices with non-zero determinant

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$$

- gp. operation = matrix multiplication $\left[(AB)_{ij} = \sum_{k=1}^{2} A_{ik}B_{kj}\right]$
- $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = identity = $I_2$

$$\rightsquigarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

cofactor matrix

Claim: $G$ is non-abelian

Eg: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightsquigarrow AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$\rightsquigarrow G = GL_n(\mathbb{K})$ for any field $\mathbb{K}$ is a non-abelian gp.

NEXT TIME: Symmetric group $S_n$ & Dihedral group $D_n$

# Basic Uniqueness Lemmas

**Lemma 1:** Neutral elements in groups are unique.

Pf/ Assume $e$, $e'$ are two neutral elements in a group $(G, *)$:

$$e \underset{\substack{\downarrow \\ e' \text{ neutral}}}{=} e * e' \underset{\substack{\downarrow \\ e \text{ neutral}}}{=} e' \qquad \square$$

**Lemma 2:** Inverses in groups are unique

Pf/ Fix $x \in G$ & write $y, y'$ for two inverses of $x$ in $G$

$$y \underset{\substack{\downarrow \\ e \text{ neutral}}}{=} y * e \underset{\substack{\downarrow \\ y' \text{ inverse}}}{=} y * (x * y') \underset{\substack{\downarrow \\ \text{Assoc}}}{=} (y * x) * y' \underset{\substack{\downarrow \\ y \text{ inverse}}}{=} e * y' \underset{\substack{\downarrow \\ y' \text{ inverse}}}{=} y' \qquad \square$$

**Lemma 3:** If $x \in G$ and $x * x = x$, then $x = e$

Pf/

$$e \underset{\substack{\downarrow \\ x^{-1} \text{ inverse}}}{=} x * x^{-1} \underset{\substack{\downarrow \\ \text{hyp.}}}{=} (x * x) * x^{-1} \underset{\substack{\downarrow \\ \text{Assoc}}}{=} x * (x * x^{-1}) \underset{\substack{\downarrow \\ x^{-1} \text{ inverse}}}{=} x * e = x \qquad \square$$

## §2. Group homomorphisms: $G, G'$ groups

**Def** A map $\varphi : G \longrightarrow G'$ is a group homomorphism if

$$\varphi(a *_G b) = \varphi(a) *_{G'} \varphi(b)$$

(algebraic structure is preserved!)

**Lemma 4**: Let $\varphi : G \longrightarrow G'$ be gp homomorphism. Then $\varphi(e) = e$ & $\varphi(x^{-1}) = (\varphi(x))^{-1}$.

Pf/ $\varphi(e) = \varphi(e * e) = \varphi(e) * \varphi(e)$

By Lemma 3 applied to $x = \varphi(e)$, we get $\varphi(e) = e' \in G'$. $\square$

⚠ This fails for monoids! (Ex: $\varphi(A) = E \ \forall A \subseteq E$ )

$\rightsquigarrow$ Def: Homomorphisms of monoids: $\varphi(a *_G b) = \varphi(a) *_{G'} \varphi(b)$ $\boxed{\&}$ $\varphi(e) = e'$