

Lecture 3 : Order of a group & Basic Isomorphism Theorems

Last time:

- Defined subgroups ($H < G$), normal subgroups ($H \triangleleft G$)
($gHg^{-1} = H \quad \forall g \in G$)
- (normal) subgroups generated by a set.
- Left cosets $G/H = \{xH : x \in G\} / \sim$ $x \sim y \Leftrightarrow x^{-1}y \in H$
- Right — $H/G = \{Hx : x \in G\} / \sim'$ $x \sim' y \Leftrightarrow xy^{-1} \in H$
- Thm: If $H \triangleleft G$, then G/H is a group under $gH * g'H = gg'H$
& $G \twoheadrightarrow G/H$ is gp hom with $\text{Ker } G = H$.
- Cyclic groups & their classification $\begin{cases} G \text{ infinite} \cong \mathbb{Z} \\ G \text{ finite} \cong \mathbb{Z}/n\mathbb{Z} \\ (n = \#G) \end{cases}$
- Hamiltonian groups (Example: Quaternions $\mathbb{Q}_8$ via gens & relations)

TODAY: Discuss order of a group & 3 Isomorphism Thms in Group Theory.

More on cosets of a gp & First counting Lemma

Def $|G| = \# \text{ elements in } G$ is called the order of G .

Eg: $|S_n| = n!$ $|\mathbb{Z}/n\mathbb{Z}| = n$.

• If $H < G$, then G breaks into a disjoint union of left cosets

$$G = \bigsqcup_{\alpha \in A} g_\alpha H$$

$A = \text{choice of representatives of } G/H$

In particular, A is in bijection with G/H . This gives us our first counting lemma.

Lemma: Assume G is finite, Then $|G| = |H| |G/H|$

Bf/ For each g $\varphi_g: H \longrightarrow gH$ is a bijection.
 $h \longmapsto gh$

Corollary: $|H|$ divides $|G|$.

Remark: $|G/H|$ is usually denoted by $(G:H) = \text{index of } H \text{ in } G$
It is possible for both G & H to be infinite & yet $(G:H) < \infty$.

Example: $G = \mathbb{Z}$ infinite but $(G:H) = 5 < \infty$
 $H = 5\mathbb{Z}$

Def: If $(G:H) < \infty$ we say H is a finite index subgroup.

• Any $g \in G$ generated a subgroup $\langle g \rangle$. So we define:

Def The order of an element g of G is the order of $H = \langle g \rangle$.

Obs.: If $|H| = n < \infty$, then $g^n = e$ ($H = \{1, g, \dots, g^{n-1}\} \cong \mathbb{Z}/n\mathbb{Z}$).
 $g^k \mapsto \bar{k}$

Corollary: $\text{Order}(g) \mid |G|$ whenever G is finite.

Exponent of a group

Def: Exponent of G = $\exp(G)$ = generator of $\{k \in \mathbb{Z} : g^k = e \ \forall g \in G\} \cap \mathbb{Z}_{\geq 0}$

Obs: • If $|G| < \infty$, then $\exp(G) < \infty$ ($g^{|G|} = e \ \forall g \in G$)

Prop: • $\exp(G) = 1 \Rightarrow G = \{e\}$

• $\exp(G) = 2 \Rightarrow G$ is abelian (Exercise)

• $\exp(G) = 3$ need not be abelian

Burnside Problem (1902): Find all $(n, m) \in \mathbb{Z}_{>0}^2$ such that if G is
[a group with m generators (minimal #) & $\exp(G) = n$, then $|G| < \infty$.

First Isomorphism Theorem

Theorem : Let G, G' be two groups and $\varphi: G \longrightarrow G'$ be a group homomorphism. Write $K = \text{Ker}(\varphi) \triangleleft G$ & $H' = \text{Im}(\varphi) < G'$.

Then we have a commutative diagram :

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ \pi \downarrow & & \uparrow i \\ G/K & \xrightarrow{\bar{\varphi}} & H' \end{array}$$

Here : π = natural projection & i is a natural inclusion.

Moreover, $\bar{\varphi}$ is an isomorphism

Remark: The other two isomorphism theorems will follow from this one.

First Iso Thm: $\frac{G}{\ker \varphi} \xrightarrow{\sim} \text{Im } \varphi$

Second Isomorphism Theorem

Warm-up:

Prop 1: Given any group homomorphism $\varphi: G_1 \rightarrow G_2$, if $H_1 < G_1$ is a subgroup, then $\varphi(H_1) < G_2$ is a subgroup.

Prop 2: For any group homomorphism: if $\varphi: G_1 \rightarrow G_2$ & $N_2 \triangleleft G_2$, then $\varphi^{-1}(N_2) \triangleleft G_1$.

Theorem: Let G be a group and $N \triangleleft G$ a normal subgroup. Then

(i) The assignment $H \rightarrow H/N$ is a bijection between

$$\left\{ \begin{array}{l} \text{Subgroups of} \\ G \text{ containing } N \end{array} \right\} \longleftrightarrow \{ \text{Subgroups of } G/N \}$$

(ii) Let $H < G$ be a subgroup containing N . Then,

H is normal if and only if H/N is normal in G/N

Furthermore, we have $\frac{G}{H} \xrightarrow{\sim} \frac{G/N}{H/N} \quad gH \mapsto gH/N$

$$\begin{array}{c}
 \text{Proof of } \left\{ \begin{array}{l} \text{Subgroups of} \\ G \text{ containing } N \end{array} \right\} \xleftrightarrow{1-\text{to-1}} \left\{ \text{Subgroups of } G/N \right\} \begin{array}{l} \xrightarrow{\pi} \\ \downarrow \\ \xleftrightarrow{\quad} \end{array} \begin{array}{l} G \\ V \\ N \subseteq H \end{array} \begin{array}{l} G/N \\ V \\ H/N \end{array}
 \end{array}$$

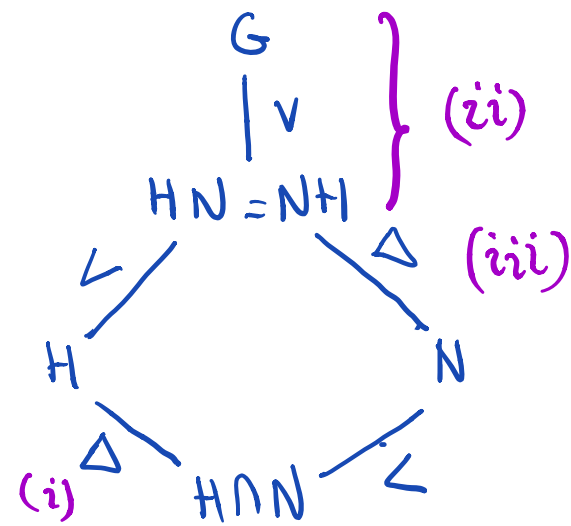
$$N \subseteq H < G : H \triangleleft G \iff H/N \triangleleft G/N$$

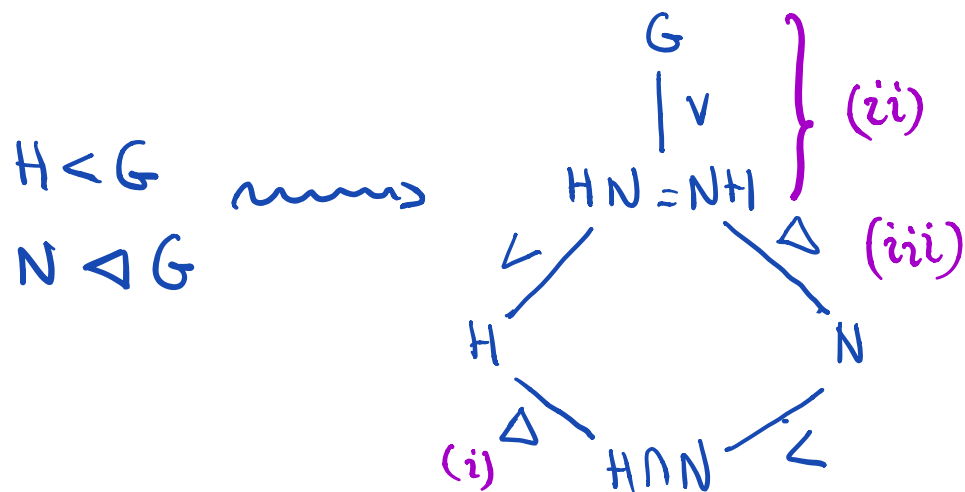
$$N \subseteq H \triangleleft G \Rightarrow G/H \xrightarrow{\sim} G/N / H/N \quad gH \mapsto gH/N$$

Third Isomorphism Theorem

Theorem: Let G be a group, $H < G$ a subgroup & $N \triangleleft G$. Then:

Cartoon encoding (i) — (iii)





$(iv) \quad \frac{H}{H \cap N} \longrightarrow \frac{HN}{N} \quad \text{is an isomorphism}$
 $h(H \cap N) \longmapsto hN$