Lecture 3: Order of a group & Basic Isomorphism Theorems

Last Time:

- Defined subgroups ($H < G$), normal subgroups ($H \triangleleft G$)
  ($gHg^{-1} = H \quad \forall g \in G$)
- (normal) subgroups generated by a set.
- Left cosets $G/H = \{xH : x \in G\}/\sim$ $\qquad x \sim y \iff x^{-1}y \in H$
- Right — $H\backslash G = \{Hx : x \in G\}/\sim'$ $\qquad x \sim' y \iff xy^{-1} \in H$
- Thm: If $H \triangleleft G$, then G/H is a group under $gH * g'H = gg'H$
  & $G \twoheadrightarrow G/H$ is gp hom with $\operatorname{Ker} G = H$.
- Cyclic groups & their classification
  - G infinite $\simeq \mathbb{Z}$
  - G finite $\simeq \mathbb{Z}/n\mathbb{Z}$ ($n = \#G$)
- Hamiltonian groups (Example: Quaternions Q_8 via gens & relations)

$$\boxed{\text{TODAY: Discuss order of a group \& 3 Isomorphisms Thms in Group Theory.}}$$

More on cosets of a gp & First counting Lemma

<u>Def</u> $|G| = \#$ elements in G is called the <u>order</u> of G.

Eg: $|S_n| = n!$ $\quad |\mathbb{Z}/n\mathbb{Z}| = n$.

- If $H < G$, then G breaks into a disjoint union of left cosets

$$\boxed{G = \bigsqcup_{\alpha \in A} g_\alpha H}$$

A = choice of representatives of G/H

In particular, A is in bijection with G/H. This gives us our first counting lemma.

<u>Lemma</u>: Assume G is <u>finite</u>. Then $|G| = |H|\,|G/H|$

Pf/ For each g $\varphi_g : H \longrightarrow gH$ is a <u>bijection</u>.
$\quad h \longmapsto gh$

<u>Corollary</u>: $|H|$ divides $|G|$.

Remark: $|G/H|$ is usually denoted by $(G:H)$ = index of H in G

It is possible for both G & H to be infinite & yet $(G:H)<\infty$.

Example: $G=\mathbb{Z}$, $H=5\mathbb{Z}$ infinite but $(G:H)=5<\infty$

Def: If $(G:H)<\infty$ we say H is a finite index subgroup.

- Any $g\in G$ generated a subgroup $\langle g\rangle$. So we define:

Def The order of an element g of G is the order of $H=\langle g\rangle$

Obs: If $|H|=n<\infty$, then $g^n=e$ $\left(H=\{1,g,\dots,g^{n-1}\}\simeq \mathbb{Z}/n\mathbb{Z}\right)$.

$$g^l \longleftarrow \bar{l}$$

Corollary: $\text{Order}(g) \mid |G|$ whenever G is finite.

Exponent of a group

Def: Exponent of G = $\boxed{\exp(G)}$ = generator of $\{k \in \mathbb{Z} : g^k = e \ \forall g \in G\} \cap \mathbb{Z}_{\geq 0}$

Obs: • If $|G| < \infty$, then $\exp(G) < \infty$ $\quad (g^{|G|} = e \ \forall g \in G)$

Prop: • $\exp(G) = 1 \implies G = \{e\}$

• ——— $= 2 \implies G$ is abelian (Exercise)

• ——— $= 3$ need not be abelian

Burnside Problem (1902). Find all $(n, m) \in \mathbb{Z}^2_{>0}$ such that if G is a group with m generators (minimal #) & $\exp(G) = n$, then $|G| < \infty$.

First Isomorphism Theorem

Theorem: Let G, G' be two groups and $\varphi : G \longrightarrow G'$ be a group homomorphism. Write $K = \mathrm{Ker}(\varphi) \lhd G$ & $H' = \mathrm{Im}(\varphi) < G'$.

Then we have a commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\quad \varphi \quad} & G' \\ {\scriptstyle \pi}\downarrow & & \uparrow{\scriptstyle i} \\ G/K & \xrightarrow[\quad \overline{\varphi} \quad]{} & H' \end{array}$$

Here: π = natural projection & i is a natural inclusion.

Moreover, $\overline{\varphi}$ is an isomorphism

Remark: The other two isomorphism theorems will follow from this one.

$$\boxed{\underline{\text{First Iso Thm:}} \quad \frac{G}{\ker\varphi} \xrightarrow{\sim} \operatorname{Im}\varphi}$$

Second Isomorphism Theorem

Warm-up:

Prop 1: Given any group homomorphism $\varphi: G_1 \to G_2$, if $H_1 < G_1$ is a subgp then $\varphi(H_1) < G_2$ is a subgroup.

Prop 2: For any group homomorphism: if $\varphi: G_1 \to G_2$ & $N_2 \lhd G_2$, then $\varphi^{-1}(N_2) \lhd G_1$.

Theorem: Let G be a group and $N \lhd G$ a normal subgroup. Then

(i) The assignment $H \longrightarrow H/N$ is a bijection between

$$\left\{ \begin{array}{c} \text{Subgroups of} \\ G \text{ containing } N \end{array} \right\} \longleftrightarrow \left\{ \text{Subgroups of } G/N \right\}$$

(ii) Let $H < G$ be a subgroup containing N. Then,

H is normal <u>if and only if</u> H/N is normal in G/N

Furthermore, we have $G/H \xrightarrow{\sim} \dfrac{G/N}{H/N}$ $\quad gH \longmapsto gH/N$

Proof of $\left\{\begin{array}{c}\text{Subgroups of}\\ G \text{ containing } N\end{array}\right\} \xleftrightarrow{\text{1-To-1}} \{\text{Subgroups of } G/N\}$

$$\begin{array}{ccc} \trianglerighteq\; G & \xrightarrow{\pi} & G/N \\ \vee & & \vee \\ N \subseteq H & \longleftrightarrow & H/N \end{array}$$

$$\boxed{N \subseteq H < G: \quad H \trianglelefteq G \iff H/N \trianglelefteq G/N}$$

$$\boxed{N \subseteq H \trianglelefteq G \Rightarrow G/H \xrightarrow{\sim} \frac{G/N}{H/N} \quad gH \longmapsto gH/N}$$

Third Isomorphism Theorem

Theorem: Let G be a group, $H < G$ a subgroup & $N \triangleleft G$. Then:

Cartoon encoding (i) — (iii)

$H < G$

$N \lhd G$

$\rightsquigarrow$

$$\begin{tikzcd}
 & G \arrow[d, dash, "\vee", "(ii)"'] & \\
 & HN = NH \arrow[dl, dash, "<"'] \arrow[dr, dash, "\Delta", "(iii)"'] & \\
H \arrow[dr, dash, "\Delta"', "(i)"] & & N \arrow[dl, dash, "<"] \\
 & H \cap N &
\end{tikzcd}$$

$$H < G \qquad N \lhd G \quad \rightsquigarrow \quad
\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash, "\vee", "\text{(ii)}"'] & \\
 & HN = NH \arrow[dl, dash, "\vee"'] \arrow[dr, dash, "\text{(iii)}\ \triangleleft"] & \\
H \arrow[dr, dash, "\text{(i)}\ \triangleleft"'] & & N \arrow[dl, dash, "<"] \\
 & H \cap N &
\end{tikzcd}$$

(iv) $\quad \dfrac{H}{H \cap N} \longrightarrow \dfrac{HN}{N}$ is an isomorphism

$\quad\quad h(H \cap N) \longmapsto hN$