Lecture 4: Group presentations by Generators & Relations

Last Time:
- order & exponent of a group
- 3 Isomorphism thms,

First Isomorphism Thm: $G/\ker\varphi \xrightarrow{\sim} \mathrm{Im}\,\varphi$ $\forall \varphi: G \to G'$ gp hom.

$\varphi: G \longrightarrow G'$ gp hom

Second Isomorphism Thm Assume $N \lhd G$ Then:

(1) $N \subseteq H < G$: $H \lhd G \iff H/N \lhd G/N$

(2) $N \subseteq H \lhd G \Rightarrow G/H \xrightarrow{\sim} (G/N)/(H/N)$ $gH \longmapsto gH/N$

$\boxed{\text{Third Isomorphism Thm}}$

$H < G$

$N \triangleleft G$ $\rightsquigarrow$

$$\begin{tikzcd}
 & G \arrow[d, dash, "\vee"] \arrow[d, phantom, shift left=6ex, "\text{(ii)}"] & \\
 & HN = NH \arrow[dl, dash, "\vee"'] \arrow[dr, dash, "\triangleleft\ \text{(iii)}"] & \\
H \arrow[dr, dash, "\text{(i)}\ \triangle"'] & & N \arrow[dl, dash, "<"] \\
 & H \cap N &
\end{tikzcd}$$

Proof of (iii).

$N \triangleleft G$ & $N < HN \leq G$

So $N \triangleleft HN$ ✓

$$\boxed{\begin{array}{llll} \text{(iv)} & H/H\cap N \longrightarrow HN/N & & \text{is an isomorphism} \\ & h(H\cap N) \longmapsto hN & & \end{array}}$$

Proof Consider

$$\begin{tikzcd}
H \arrow[r, hook, "i", "\text{inclusion}"'] \arrow[rr, bend left, "\varphi = \pi \circ i"] & HN \arrow[r, two heads, "\pi", "\text{projection}"'] & HN/N
\end{tikzcd}$$

• φ gp homomorph.

Claim 1: φ is surjective

Pf/ $hxN = hN$ for $h \in H$, $x \in N$

But $hN = i(h)$

$\Rightarrow \varphi(h) = hxN$

Claim 2 $\operatorname{Ker}\varphi = H \cap N$

Pf/ $h \in \operatorname{Ker}\varphi \iff \varphi(h) = \bar{e} \in HN/N$

$(h \in H)$ $\iff hN = N \iff h \in H \cap N$ ✓

$\xrightarrow{\text{1st Iso Thm}}$ $\boxed{H/H\cap N \simeq HN/N}$

First Isomorphism Thm

Second interpretation: Fix $f : G \twoheadrightarrow G'$ surjective gp hom. & $H \trianglelefteq G$ with $H \subseteq \operatorname{Ker} f$. Then $H = \operatorname{Ker} f \iff G/H \overset{\bar{F}}{\cong} G'$

Pf/ $(\Rightarrow)$ 1^{st} Isomorphism Thm

$(\Leftarrow)$ Consider

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & G' \\ G/H \arrow[ur, "\bar{F}\ \text{iso}"'] & \end{tikzcd}$$

$\bar{F}$ iso $\iff$ $\bar{F}$ inj.

$\operatorname{ker}(f)/H = \operatorname{ker}(\bar{F}) = e_{G/H} = eH$

Example $G = \text{Free}(2) = \langle a, b \mid \text{no relations} \rangle$ with concatenation + cancellation

Take $\varphi : G \longrightarrow \mathbb{Z}^2$, $w \longmapsto (\#a\text{'s}, \#b\text{'s})$

- φ surj ✓
- φ group hom ✓

Ex: $\varphi(a^2 b^2 a^{-7}) = (2-7, 2) = (-5, 2) = \varphi(a^{-5} b^2) = \varphi(b^2 a^{-5})$.

- $H = \langle x y x^{-1} y^{-1} \mid x, y \in G \rangle = [G : G] \trianglelefteq G$ [HW1] & $\subseteq \operatorname{Ker} \varphi$
- $G/H = \langle a, b \mid ab = ba \rangle \overset{\bar{\varphi}}{\cong} \mathbb{Z}^2$
  $a^m b^n \longmapsto (m, n)$

$\underset{\text{Thm}}{\Longrightarrow}$ $\boxed{H = \operatorname{Ker} \varphi}$

TODAY: Group Presentations

Q: How to describe a group?

A Many options: (1) Symmetries of a set (bijections $X \longrightarrow X$)

(2) Multiplication Table (eg Q_8)

→ (3) Generators & relations.

Advantages: (1) & (3) Associativity is automatic.

Disadvantage: (3) Presentation is not unique & can get trivial sp from a complicated presentation (Example later on)

Word problem (Dehn 1911) Algorithm to decide if 2 words on gens of G give the same element in group.

Free Groups

Definition: Given a set A, let $\underline{\mathrm{Free}(A)} = \{\text{words on } A\}$ with operation = concatenation & cancellation.

$\mathrm{Free}(\emptyset) = \{e\}$ (empty word)

Obs: If $w \in \mathrm{Free}(A)$, then w has a unique expression of the form

$$w = x_1^{n_1} x_2^{n_2} \cdots x_\ell^{n_\ell} \quad [\ell = \mathrm{length}(w)] \quad \text{where}$$

$$\begin{cases} \cdot\, x_1, x_2, \ldots, x_\ell \in A, \quad x_1 \neq x_2,\ x_2 \neq x_3, \ldots, x_i \neq x_{i+1}, \ldots, x_{\ell-1} \neq x_\ell \\ \cdot\, n_1, \ldots, n_\ell \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Convention: $\ell = 0 \iff w = e \in \mathrm{Free}(A)$; **Note** $w^{-1} = x_\ell^{-n_\ell} \cdots x_1^{-n_1}$

Q: What would it take to define a group homomorphism

$f : \mathrm{Free}(A) \longrightarrow H$ for an arbitrary group H?

A: ① Specify $f(a) \in H \ \forall a \in A$ (free will, nothing to check!)

② $w \in \mathrm{Free}(A) \rightsquigarrow w = x_1^{n_1} x_2^{n_2} \cdots x_\ell^{n_\ell}$ uniquely!

$\implies f(w) = f(x_1)^{n_1} f(x_2)^{n_2} \cdots f(x_\ell)^{n_\ell}$ is the only possible defn! (unambiguous)

Corollary $\left\{\begin{array}{c}\text{Group homomorphisms} \\ \mathrm{Free}(A) \longrightarrow H\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Set Maps} \\ A \longrightarrow H\end{array}\right\}$

Universal Property of Free (A) :

Given any set map $f : A \longrightarrow H$, there exists a unique group homomorphism $\tilde{f} : \mathrm{Free}(A) \longrightarrow H$ satisfying

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathrm{Free}(A) \arrow[d, dashed, "\exists ! \tilde{f}"] \\ & H \end{tikzcd}$$

In particular Free (A) is unique up to unique isomorphism.

$A \rightsquigarrow$ generators of a group G. Relations = ?

$\boxed{\text{Relations} \subseteq \text{Free}(A)}$

Recall: $X \subseteq G$ gp $\rightsquigarrow$ $N(X)$ = smallest normal subgp of G containing X
(Lecture 2)

$$= \bigcap_{\substack{N \trianglelefteq G \\ X \subseteq N}} N$$

Def: Given a set $\underline{\underline{A}}$ (generators) & $R \subseteq \text{Free}(A)$ (relations), we

define $\boxed{\langle A \mid R \rangle := \text{Free}(A) \big/ N(R)}$ (want a group!)

Q: What would it take to define a group hom. $f: \langle A \mid R \rangle \longrightarrow H$?

A ① Specify $f(a) \in H \quad \forall a \in A$

$\rightsquigarrow \tilde{f} : \text{Free}(A) \longrightarrow H$ gp homomorphism

② Make sure $\tilde{f}(r) = 0 \quad \forall r \in R \subset \text{Free}(A)$

EXAMPLES

Ex 1: $G = \langle x, y \mid xy^2 = y^3x,\ yx^2 = x^3y \rangle \simeq \{e\}$

Why? $xy^2 = y^3x \implies xy^4 = xy^2y^2 = y^3xy^2 = y^6x$

$\leadsto xy^8 = xy^4y^4 = y^6xy^4 = y^{12}x$

$\leadsto x^2y^8 = xy^{12}x = xy^8y^4x = y^{12}xy^4x = y^{12}y^6x^2$

So $\boxed{x^2y^8x^{-2} = y^{18}}$ • Similarly, $\boxed{x^3y^8x^{-3} = y^{27}}$

(Indeed $x^3y^8x^{-3} = x\,x^2y^8x^{-2}x^{-1} = xy^{18}x^{-1} = xy^8y^{10}x^{-1} =$
$= y^{12}xy^8y^2x^{-1} = y^{12}y^{12}xy^2x^{-1} = y^{24}y^3xx^{-1} = y^{27}$ $\square$)

• But 2^{nd} relation gives $yx^2y^{-1} = x^3$, thus:
$y^{27} = x^3y^8x^{-3} = yx^2y^{-1}y^8yx^{-2}y^{-1} = yx^2y^8x^{-2}y^{-1} = y^{18}$ $\Big\} \implies \boxed{y^9 = e}$

$\implies e = x^{-1}y^9x = (x^{-1}y^3x)^3 = (x^{-1}xy^2)^3 = y^6 \leadsto \boxed{y^3 = e}$

$\leadsto$ 1^{st} Rel gives $y^2 = e$, so $\boxed{y = e}$. Thus, 2^{nd} Reln gives $\boxed{x = e}$

Ex 2 $\quad G = \mathrm{Free}(\{a,b\})$, $\quad \varphi : \mathrm{Free}(\{a,b\}) \twoheadrightarrow \mathbb{Z}^2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad w \longmapsto (\#a\text{'s}, \#b\text{'s})$

- $H = \mathrm{Ker}\,\varphi \trianglelefteq G$, $\quad aba^{-1}b^{-1} \in H$

Set $R = \{aba^{-1}b^{-1}\} \in \mathrm{Free}(A) \rightsquigarrow N(R) \subseteq H$

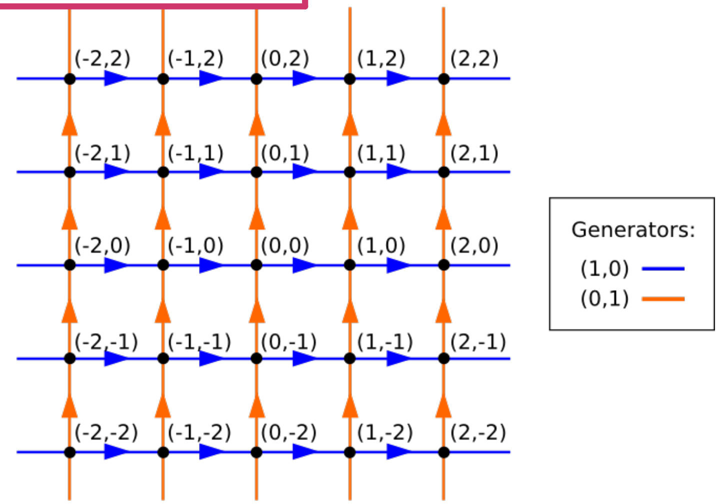
Claim: $\langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle \overset{\bar{\varphi}}{\cong} \mathbb{Z}^2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a^k b^m \longmapsto (k, m).$

Conclude: $N(R) = H = \mathrm{ker}\,\varphi.$

Obs:
- Smallest subgroup containing $x = aba^{-1}b^{-1}$ is $\langle x \rangle \cong \mathbb{Z}.$
- Smallest <u>normal</u> subgp containing x is $\mathrm{Ker}\,\varphi$. This is not even finitely generated!

$\boxed{K = N(aba^{-1}b^{-1}) \text{ is not f.g.}}$



Pf/ View gens of Free($\{a,b\}$) inside $\mathbb{Z}^2$ via φ, ie

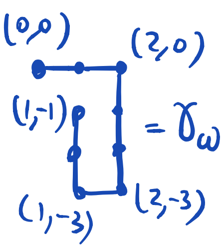
$$a \longleftrightarrow (1,0)$$

$$b \longleftrightarrow (0,1)$$

- Each $w \in$ Free($\{a,b\}$) defines a path γ_w in $\mathbb{R}^2$ starting at $(0,0)$ by reading w from left to right, ie: a^k : move k steps along x-axis $\begin{cases} k \geq 0 \rightsquigarrow \text{right move} \\ k<0 \rightsquigarrow \text{left} \end{cases}$ & b^k : move k steps along y-axis $\begin{cases} k \geq 0 \rightsquigarrow \text{upwards} \\ k \leq 0 \rightsquigarrow \text{downwards} \end{cases}$

Ex: $w = a^2 b^{-3} a^{-1} b^2$

γ_w :



$\Longrightarrow$ Define a map $d : \text{Free}\{a,b\} \longrightarrow \mathbb{R}_{\geq 0}$

$$\boxed{d(w) = \max\{\text{distance}(\underline{0}, p) \mid p \in \gamma_w\}}$$

Remarks (1) $\forall\, w \in H$ Endpoint of $w = \underline{0}$.

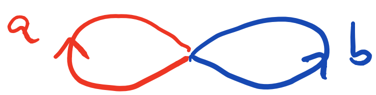
(2) If $w_1, w_2 \in H \Rightarrow d(w_1 w_2) \leq \max\{d(w_1), d(w_2)\}$

(3) $d(\underbrace{a^n b a^{-n} b^{-1}}_{w = [a^n : b]}) = \text{distance}\{(0,0), (n,1)\} = \sqrt{n^2+1} \xrightarrow[n\to\infty]{} \infty$

$\Longrightarrow$ K not f.g.

<u>Obs</u>: $\exists$ Alternative Proof via Algebraic Topology.

- $\mathrm{Free}(\{a,b\})$ = fundamental group of bouquet of 2 S^1's:



- $\tilde{X}$ = universal cover
- $F_2' = [\mathrm{Free}(\{a,b\}), \mathrm{Free}(\{a,b\})]$

Reidemeister-Schreier Thm:

$\Downarrow$

F_2' cannot be finitely generated.

- This Topological proof leads to a general Theorem:

Thm: Fix G an infinite group & $\Phi: \mathrm{Free}(n) \twoheadrightarrow G$ gp homomorph.

Then: $\ker(\Phi)$ is trivial or it is not finitely generated.

Example 3: $D_n = \{1, \rho, \rho^2, \dots, \rho^{n-1}, s, s\rho, \dots, s\rho^{n-1}\}$ **Dihedral Gp**

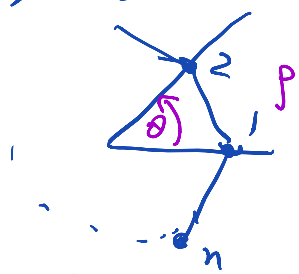
(symmetries of reg n-gon)

Generators $= \{s, \rho\}$

Relations: $s^2 = \rho^n = e, \quad s\rho s = \rho^{-1} \quad (\Leftrightarrow (s\rho)^2 = e)$

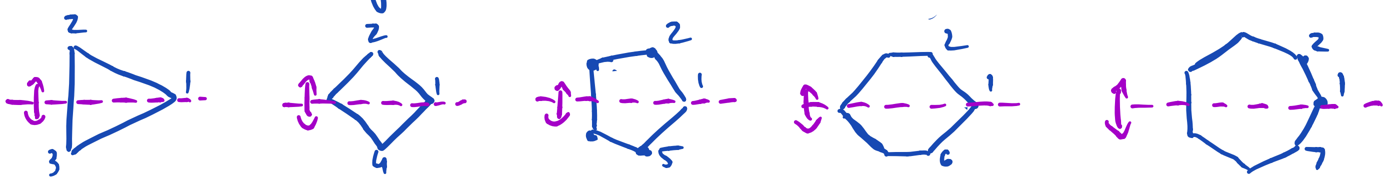
Claim: $D_n = \langle s, \rho \mid s^2, \rho^n, (s\rho)^2 \rangle$

View • ρ = rotation of angle $\frac{2\pi}{n} = \theta$



• s = reflection along x-axis

Ex



$n=3$ $\quad$ $n=4$ $\quad$ $n=5$ $\quad$ $n=6$ $\quad$ $n=7$

viewed in S_n: $(2\ 3)$ $\quad$ $(2\ 4)$ $\quad$ $(2\ 5)(3\ 4)$ $\quad$ $(2\ 6)(3\ 4)$ $\quad$ $(2\ 7)(3,6)(4,5)$

In general: $s \longleftrightarrow (2\ n)(3\ \ n-1) \cdots \left(\frac{n}{2}-1, \frac{n}{2}+1\right)$ $\quad$ n even

$\qquad (2\ n)(3\ \ n-1) \cdots \left(\frac{n+1}{2}, \frac{n+1}{2}+1\right)$ $\quad$ n odd

• $|D_n| = 2n$ & relations hold in $D_n \Rightarrow$ No other relations need to be added!

Lemma: There exists a group homomorphism (sign representation)

$$f: D_n \longrightarrow \{\pm 1\} \quad \text{with} \quad f(s) = -1 \quad f(\rho) = 1$$

Pf/ $D_n = \langle s, \rho \mid s^2 = \rho^n = (s\rho)^2 = e \rangle$

Write $\varphi: \mathrm{Free}(s,\rho) \twoheadrightarrow \{\pm 1\}$ with $f(s) = f(\rho) = 1$

Want to factor this map through D_n i.e. $f = \overline{\varphi}: D_n \longrightarrow \{\pm 1\}$

To define the map $\overline{f}$ we need to check: $f(s) = -1$ $f(\rho) = 1$ preserves the relations $f(s^2) = f(\rho^n) = f((s\rho)^2) = 1$

but this is clear. $\square$

- $\mathrm{Ker}\, f = \{$ words in s, ρ of even length$\}$

$$= \{e, \rho, \rho^2, \ldots, \rho^{n-1}\} =: K \simeq \mathbb{Z}/n\mathbb{Z}$$

- $\mathrm{Im}\, f = \{\pm 1\}$, so f is surjective.

Conclusion: $D_n/K \simeq \{\pm 1\}$ by 1st Iso Thm.

Example 4: D_n can be generated by 2 reflections (π involutions)

$$D_n = \langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^2 = (\sigma_1\sigma_2)^n = e \rangle$$

How? $\sigma_1 \leftrightarrow s$ $\quad$ ($\sigma_1\sigma_2 = ss\rho = \rho$ so relation holds!)

$\sigma_2 \leftrightarrow s\rho$ $\quad$ $\sigma_2^2 = s\rho s\rho = \rho^{-1}\rho = e$

This is an example of a Coxeter Group. (1934)

Def: A Coxeter group has presentation $\langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} \rangle$,

where $m_{ij} = \begin{cases} 1 & i = j \\ \geq 2 & i \neq j \end{cases}$ $\quad$ ($m_{ij} = \infty$ means no relation between r_i & r_j is imposed)

$m_{ij} \in \mathbb{N} \cup \{\infty\}$

- $(xy)^2 = e$ means $xy = yx$
- Avoid redundancies by writing $m_{ij} = m_{ji}$

If $y^2 = 1$ & $(xy)^m = 1$ then $(yx)^m = (yx)^m yy = y(xy)^m y = y^2 = 1$

Classification of finite Coxeter groups (1935)

1. Correspond to Coxeter-Dynkin Diagrams
2. They can be realized as reflections of finite-dimensional Euclidean spaces

Example 5: S_n is generated by transpositions $\sigma_{ij} = (i, j)$ $1 \le i < j \le n$

$\binom{n}{2}$ many!

FACT 1: Any permutation is a product of disjoint cycles (in any order)

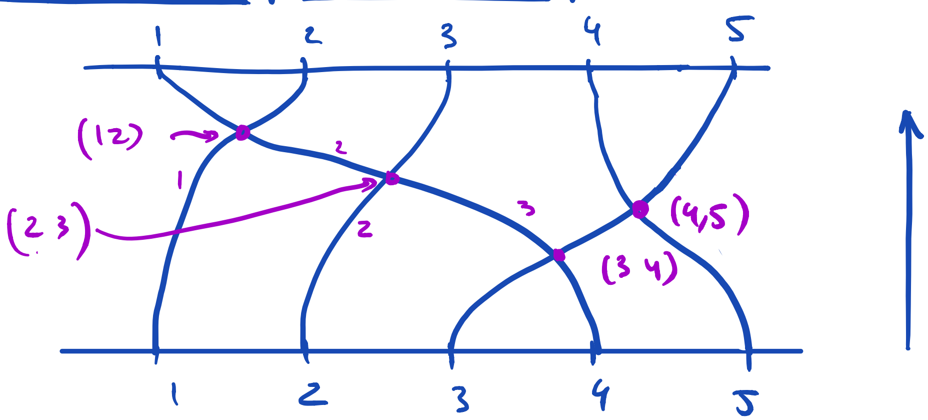
Eg: $\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 & 5 & 4 \end{array} = (123)(45) = (45)(123)$

FACT 2: Any cycle is a product of transpositions

Pf/ $(i_1\ i_2, \ldots i_k) = (i_1\ i_2)(i_2\ i_3) \cdots\cdots (i_{k-1}\ i_k)$

• More is true! $\boxed{S_n = \langle \sigma_{i,i+1} \quad 1 \le i \le n-1 \rangle}$ simple transpositions $(n-1)$ many!

• Pictorial Proof in one example: $(14532) = (34)(45)(23)(12)$



read transpositions in this order.

(In general, use "slide" strings so that we only transpose consecutive strings)

Formal Proof: By Facts 1 & 2, can reduce to transpositions σ_{ij}

- $j = i+1$, nothing to show.
- If $j > i+1$, use:

$$(i\ j) = (j-1\ j)\boxed{(i\ j-1)}(j-1, j)$$

$\uparrow$ induct.

Q: Relations among $s_i = (i\ i+1)$?

$s_i^2 = e$ ✓ ; $s_i s_j = s_j s_i$ if $|j-i| > 1$

$(s_i s_{i+1}) = (i\ i+1)(i+1, i+2) = (i\ i+1\ i+2)$ 3-cycle

So $(s_i s_{i+1})^3 = e$, so $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Define $$\mathcal{P}_n = \left\langle a_1, \dots, a_{n-1} \;\middle|\; \begin{array}{l} a_i^2 = e \quad \forall i \\ a_i a_j = a_j a_i \quad \text{if } |i-j| > 1 \\ a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \end{array} \right\rangle$$

Prop $\mathcal{P}_n \simeq S_n$

Pf/ Above calculation yields $\mathcal{P}_n \xrightarrow{\varphi} S_n$ (onto), $a_i \longmapsto s_i \ \forall i$. To show $\ker \varphi = N\langle \text{relations} \rangle$ we'll need group actions (next time!)