Lecture 5: Group Actions on Sets

So far: (1) Defined useful terms from Group Theory:

- Group, subgroup, subgroup generated by a subset, order & exponent
- Normal subgroup, normal subgp ——————
- Left / Right cosets (G/H & $H\backslash G$), quotient groups
- Group homomorphisms / Isomorphisms, Kernel & Image of gp hom.
- Free group, Generators & relations; Examples (Free(A), S_n, D_n)

(LAST TIME)

(2) Main Results: 3 Isomorphism Thms, Classification of cyclic gps.

TODAY: Groups acting on sets

Group Actions on Sets

Def: Let G be any group and let X be a set. A (left) action of G on X is a set map

$$\begin{aligned} G \times X &\xrightarrow{\alpha} X \\ (g, x) &\longmapsto \alpha(g,x) =: g \cdot x \end{aligned}$$

satisfying   NOTATION $G \circlearrowright X$

(i) $e \cdot x = x \quad \forall x \in X$ (ii) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \quad \forall g_1, g_2 \in G \ \forall x \in X$

[For a right action use (ii)' $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ NOTATION: $X \circlearrowleft G$]

Observation: If $G \circlearrowright X$, then each $g \in G$ defines a set map:

$$\begin{aligned} \tau(g) := \alpha(g, -) : X &\longrightarrow X \\ x &\longmapsto g \cdot x \end{aligned}$$

<u>Example</u> $G = GL_n(\mathbb{R}) \circlearrowright X = \mathbb{R}^n$ by

$$G \times X \xrightarrow{\alpha} X \qquad \text{matrix multiplication}$$
$$(A, \underline{x}) \longmapsto A\underline{x}$$

<u>Def</u>: The <u>orbit</u> of an element $x \in X$ is the following subset of X

$$\boxed{G \cdot x} := \{ g \cdot x \mid g \in G \} \subseteq X$$

<u>Def</u> The <u>stabilizer</u> of an element $x \in X$ is the following subgroup of G

$$\boxed{\mathrm{Stab}_G(x)} := \{ g \in G \mid g \cdot x = x \} \leq G$$

<u>Obs</u>: Stab_G need <u>not</u> be a normal subgroup

<u>Def</u>: The <u>fixed point set</u> of an element $g \in G$ is

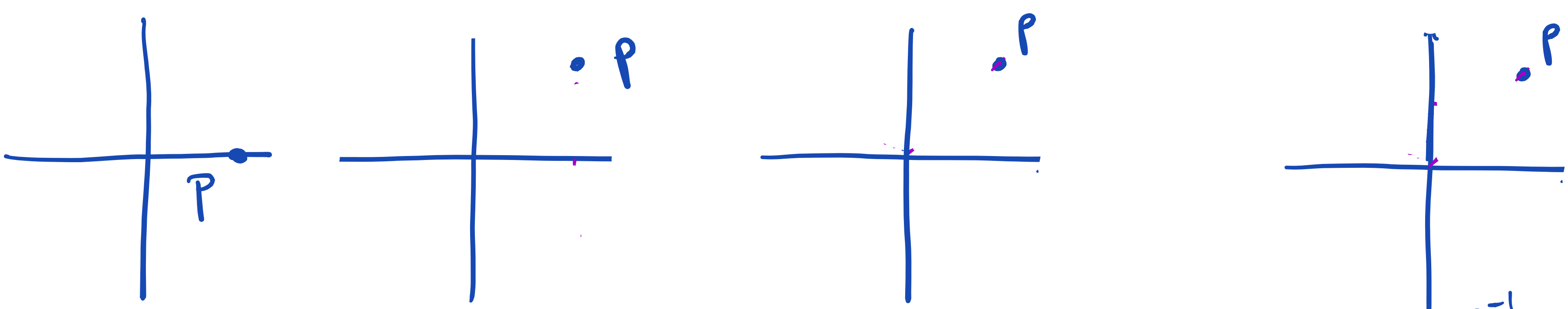
$$\boxed{X^g} := \{ x \in X \mid g \cdot x = x \} \subseteq X$$

Example: $D_n \hookrightarrow GL_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{R}^2)$

$$s \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(reflection about x-axis)}$$

$$\rho \longmapsto \begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix} \quad \left(\text{rotation of angle } \theta = \frac{2\pi}{n}\right)$$

- This is a group homomorphism, so it defines an action $D_n \circlearrowright \mathbb{R}^2$.
- $\mathrm{Orbit}((0,0)) = (0,0)$ Q: $\mathrm{Orbit}(p)$ for $p \neq (0,0)$



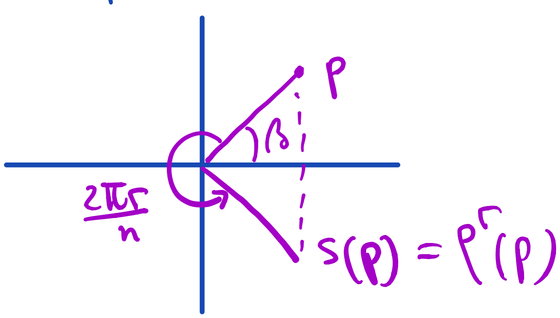
$s\rho = \rho^{-1}s$

$\text{Orbit}(p) \supseteq \{p, \rho(p), \cdots, \rho^{n-1}(p)\}$ ⇝ n many (*)

Claim: $|D_n \cdot p| = 2n \iff s(p) \notin \{p, \rho(p), \cdots, \rho^{n-1}(p)\}$

Pf/

• If $s(p) = \rho^r(p)$ for some $r = 0, \cdots, n-1$ then:



Q: $\text{Stab}_{D_n}(p) = ?$

Proposition: Let $G \circlearrowright X$

(1) For every $x \in X$ we have a (set) bijection

$$G/\mathrm{Stab}_G(x) \longrightarrow G \cdot x$$

(2) For every $\sigma \in G$ and $x \in X$, we have an isomorphism of groups

$$\begin{aligned} \mathrm{Stab}_G(x) &\longrightarrow \mathrm{Stab}_G(\sigma \cdot x) \\ g &\longmapsto \sigma g \sigma^{-1} \end{aligned}$$

(Conjugation by σ)

Proof:

Properties of Group Actions

① **Free** We say a G-action on X is free if $g \cdot x = x$ for some x $\Rightarrow g = e$

Equivalently . $\mathrm{Stab}_G(x) = \{e\}$ $\forall x \in X$

. If G is finite, then all orbits have the same size $= |G|$

②. **Transitive**: We say a G-action on X is transitive if $\forall x, y \in X$, $\exists g \in G$ such that $g \cdot x = y$

Equivalently: $G \cdot x = X$ for all $x \in X$. (only one orbit)

③ **Faithful**: We say a G-action on X is faithful if $G \xrightarrow{\sigma} \mathrm{Aut}_{Set}(X)$ is injective (G is faithfully represented in $\mathrm{Aut}_{Set}(X)$)

Equivalently: $g \cdot x = x$ $\underline{\forall} x \in X \Rightarrow g = e$

Obs: Free $\Rightarrow$ Faithful but Faithful $\not\Rightarrow$ Free

$\boxed{\underline{\text{Free}} = \mathrm{Stab}_G(x) = e \ ; \ \underline{\text{Transitive}} = 1 \text{ orbit} ; \underline{\text{Faithful}} \ G \hookrightarrow \mathrm{Aut}(X)}$

<u>Examples</u>: ① $D_n \circlearrowright \mathbb{R}^2, \{(0,0)\}$

- Faithful
- Free
- Transitive

② $S_n \circlearrowright \{1, 2, \dots, n\}$

- Faithful
- Free
- Transitive

③ $G \circlearrowright G/H$ by $g(g'H) = gg'H$

Faithful ?
Free ?
Transitive ?

(Exercise)

Counting Lemmas

Def: $x \sim_G x'$ in X iff $\exists g \in G$ $g \cdot x = x'$

$\Longrightarrow G \backslash X = X/\sim_G$ equiv classes = orbits

Easy Observation:

$$X = \bigsqcup_{\alpha \in G\backslash X} G \cdot x_\alpha \Longrightarrow |X| = \sum_{\alpha \in G\backslash X} |G \cdot x_\alpha|$$

orbit representative

Recall (Prop): $G/\mathrm{Stab}_G(x) \xrightarrow{\text{bij}} G \cdot x$ & $\mathrm{Stab}_G(x) \xrightarrow{\text{conj } \sigma} \mathrm{Stab}_G(\sigma \cdot x)$

Corollary: (a) $|G| = |G \cdot x|\,|\mathrm{Stab}_G x| \quad \forall x \in X$

(b) $$|X| = \sum_{\alpha \in G\backslash X} |G| / |\mathrm{Stab}_G(x_\alpha)|$$

$$\boxed{\textbf{Burnside' Lemma}: \quad |G\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^g|}$$

Pf/

$$\boxed{\underline{\text{Application}}:\ p \text{ prime}, p>0,\ m\in\mathbb{Z}_{\geq 1} \implies \binom{p^r m}{p^r} \equiv m \mod (p)}$$

Pf/