Lecture 5: Group Actions on Sets

So far: (1) Defined useful terms from Group Theory:

- Group, subgroup, subgroup generated by a subset, order & exponent
- Normal subgroup, normal subgp ———————
- Left / Right cosets (G/H & $H\backslash G$), Quotient groups
- Group homomorphisms / Isomorphisms, Kernel & Image of gp hom.
- Free group, Generators & relations ; Examples (Free(A), S_n, D_n)

(LAST TIME)

(2) Main Results: 3 Isomorphism Thms, Classification of cyclic gps.

TODAY: Groups acting on sets

Group Actions on Sets

Def: Let G be any group and let X be a set. A (left) <u>action of G on X</u> is a set map

$$\begin{aligned} G \times X &\xrightarrow{\alpha} X \\ (g, x) &\longmapsto \alpha(g,x) =: g \cdot x \end{aligned}$$

satisfying

<u>NOTATION</u> $G \circlearrowright X$

(i) $e \cdot x = x \quad \forall x \in X$ (ii) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \quad \forall g_1, g_2 \in G \ \forall x \in X$

[For a <u>right action</u> use (ii)' $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ <u>NOTATION</u>: $X \circlearrowleft G$]

<u>Observation</u>: If $G \circlearrowright X$, then each $g \in G$ defines a set map:

$$\begin{aligned} \tau(g) := \alpha(g, -) : X &\longrightarrow X \\ x &\longmapsto g \cdot x \end{aligned}$$

It satisfies:

$$\left.\begin{aligned} &\text{(i)} \quad \tau(e) = Id_X \\ &\text{(ii)} \quad \tau(g_1 g_2)(x) = g_1 g_2 \cdot x = \tau(g_1) \circ \tau(g_2)(x) \end{aligned}\right\} \Rightarrow \begin{aligned} \tau(g_1^{-1}) \circ \tau(g_1) &= \tau(g_1^{-1} g_1) = Id_X \\ \tau(g_1) \circ \tau(g_1^{-1}) &= \tau(g_1 g_1^{-1}) = Id_X \end{aligned}$$

$$\boxed{\text{Conclusion: } \tau : G \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(X) := \{ f : X \to X \text{ bijection} \} \text{ is a gp hom.}}$$

Example $G = GL_n(\mathbb{R}) \circlearrowright X = \mathbb{R}^n$ by matrix multiplication

$$G \times X \xrightarrow{\alpha} X$$
$$(A, \underline{x}) \longmapsto A\underline{x}$$

Def: The <u>orbit</u> of an element $x \in X$ is the following subset of X

$$\boxed{G \cdot x} := \{ g \cdot x \mid g \in G \} \subseteq X$$

Def The <u>stabilizer</u> of an element $x \in X$ is the following subgroup of G

$$\boxed{\mathrm{Stab}_G(x)} := \{ g \in G \mid g \cdot x = x \} \leq G$$

Obs: Stab_G need <u>not</u> be a normal subgroup $\left(S_{n-1} = \mathrm{Stab}_{S_n}(\{n\}) \ntrianglelefteq S_n \right)$

Def: The <u>fixed point set</u> of an element $g \in G$ is

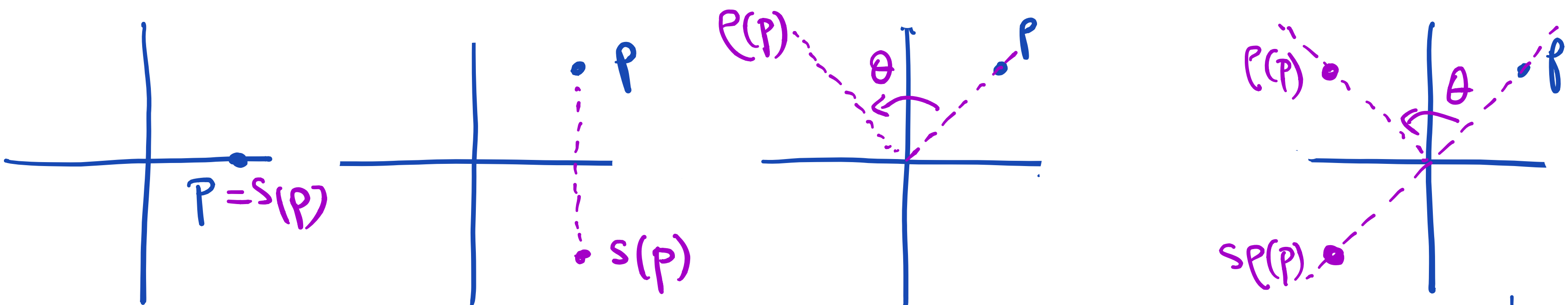
$$\boxed{X^g} := \{ x \in X \mid g \cdot x = x \} \subseteq X$$

Example: $D_n \hookrightarrow GL_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{R}^2)$

$s \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (reflection about x-axis)

$\rho \longmapsto \begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix}$ (rotation of angle $\theta = \frac{2\pi}{n}$)

- This is a group homomorphism, so it defines an action $D_n \circlearrowright \mathbb{R}^2$.
- $\mathrm{Orbit}((0,0)) = (0,0)$ Q: $\mathrm{Orbit}(p)$ for $p \neq (0,0)$



$s\rho = \rho^{-1}s$

Obs: $\boxed{\left|\{p, \rho(p), \ldots, \rho^{n-1}(p)\}\right| = n.}$ $(*)$

$$\mathrm{Orbit}(p) = \{p, \rho(p), \rho^2(p), \ldots, \rho^{n-1}(p), s(p), s\rho(p), \ldots, s\rho^{n-1}(p)\}$$
$$\supseteq \{p, \rho(p), \ldots \rho^{n-1}(p)\}$$

$$\text{Orbit}(p) \supseteq \{p, \rho(p), \dots, \rho^{n-1}(p)\} \rightsquigarrow n \text{ many } (*)$$

Claim: $|D_n \cdot p| = 2n \iff s(p) \notin \{p, \rho(p), \dots, \rho^{n-1}(p)\}$

Pf/ $|D_n \cdot p| < 2n \iff$ there is a repeated element x.

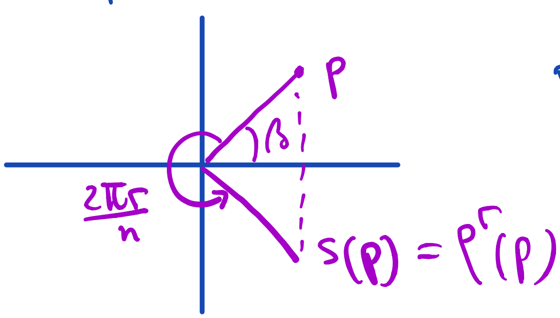
Now $x = s\rho^k(p)$ or $\rho^j(p)$ for some k, j

BUT $s\rho^k(p) \neq s\rho^j(p)$ & $\rho^k(p) \neq \rho^j(p)$ by $(*)$

So only option is $x = \underbrace{s\rho^k}_{}(p) = \rho^j(p) \iff s(p) = \rho^{k+j}(p)$ for some k, j. □

$= \rho^{-k} s(p)$

• If $s(p) = \rho^r(p)$ for some $r = 0, \dots, n-1$ then:



$$p = Re^{-\beta i} = Re^{(\beta + \frac{2\pi r}{n})i} \Rightarrow \beta + \frac{2\pi r}{n} \equiv -\beta \quad (2\pi)$$

$$\beta \equiv -\frac{\pi r}{n} \quad (\pi)$$

So $p = R\begin{bmatrix} \cos\frac{\pi r}{n} \\ -\sin\frac{\pi r}{n} \end{bmatrix}$ & $|D_n \cdot p| = n$.

Q: $\text{Stab}_{D_n}(p) = ?$ $\quad s\rho^j(p) = \rho^{-j}s(p) = \rho^{r-j}(p) = p \iff j = r$

• $\rho^j(p) \neq p \quad \forall j = 1, \dots, r$

$\Rightarrow \boxed{\text{Stab}_{D_n}(p) = \{e, s\rho^r\}}$

Proposition: Let $G \curvearrowright X$

(1) For every $x \in X$ we have a (set) bijection

$$G/\mathrm{Stab}_G(x) \longrightarrow G\cdot x$$

(2) For every $\sigma \in G$ and $x \in X$, we have an isomorphism of groups

$$\begin{aligned} \mathrm{Stab}_G(x) &\longrightarrow \mathrm{Stab}_G(\sigma\cdot x) \\ g &\longmapsto \sigma g \sigma^{-1} \end{aligned}$$

(Conjugation by σ)

Proof: (1) Define $f := \alpha(-,x) : G \twoheadrightarrow G\cdot x$, $g \longmapsto g\cdot x$

But $g\cdot x = h\cdot x \iff h^{-1}g\cdot x = x$
$\iff h^{-1}g \in \mathrm{Stab}_G(x)$

$\implies$
$$\begin{tikzcd} G \arrow[r, two heads, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G\cdot x \\ G/\mathrm{Stab}_G(x) \arrow[ur, "\bar F", "\sim"'] & \end{tikzcd}$$

$\bar F$ is a bijection

$\bar F(g\,\mathrm{Stab}_G) = g\cdot x = f(g)$

(2) • $g \in \mathrm{Stab}_G(x) \iff g\cdot x = x$
$\iff (\sigma g \sigma^{-1})(\sigma x) = \sigma x$
$\iff \sigma g \sigma^{-1} \in \mathrm{Stab}_G(\sigma x)$

• Conj_σ is iso on G

$\square$

Properties of Group Actions

① **Free** We say a G-action on X is free if $g \cdot x = x$ for some x $\Longrightarrow g = e$

Equivalently . $Stab_G(x) = \{e\}$ $\forall x \in X$

. If G is finite, then all orbits have the same size $= |G|$

②. **Transitive**: We say a G-action on X is transitive if $\forall x, y \in X$, $\exists g \in G$ such that $g \cdot x = y$

Equivalently: $G \cdot x = X$ for all $x \in X$. (only one orbit)

③ **Faithful**: We say a G-action on X is faithful if $G \xrightarrow{\sigma} Aut_{Set}(X)$ is injective (G is faithfully represented in $Aut_{Set}(X)$)

Equivalently: $g \cdot x = x \ \underline{\forall} x \in X \Longrightarrow g = e$

Obs: Free $\Longrightarrow$ Faithful but Faithful $\not\Longrightarrow$ Free

$\boxed{\underline{\text{Free}} = \mathrm{Stab}_G(x) = e \; ; \; \underline{\text{Transitive}} = 1 \text{ orbit} ; \underline{\text{Faithful}} \; G \hookrightarrow \mathrm{Aut}(X)}$

Examples: ① $D_n \circlearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

- Faithful ✓
- Free ✗ ($\exists$ orbits of size $n \neq |D_n|$)
- Transitive ✗ (there are many orbits!)

② $S_n \circlearrowright \{1, 2, \ldots, n\}$

- Faithful ✓ $S_n = \mathrm{Aut}(\{1, \ldots, n\})$
- Free ✗ (S_{n-1} fixes $\{n\}$)
- Transitive ✓ (induct on n)

③ $G \circlearrowright G/H$ by $g(g'H) = gg'H$

Faithful ?
Free ?
Transitive ?

(Exercise)

Counting Lemmas

Def: $x \sim_G x'$ in X iff $\exists g \in G$ $g \cdot x = x'$
$\Longrightarrow G\backslash X = X/\sim_G$ equiv classes = orbits

Easy Observation:

$$X = \bigsqcup_{\alpha \in G\backslash X} G \cdot x_\alpha \Longrightarrow |X| = \sum_{\alpha \in G\backslash X} |G \cdot x_\alpha|$$

orbit representative

Recall (Prop)

$$G/\mathrm{Stab}_G(x) \xrightarrow{\text{bij}} G\cdot x \quad \& \quad \mathrm{Stab}_G(x) \xrightarrow{\text{conj } \sigma} \mathrm{Stab}_G(\sigma\cdot x)$$

Corollary:

(a) $|G| = |G\cdot x|\,|\mathrm{Stab}_G x| \quad \forall x \in X$

(b) $$|X| = \sum_{\alpha \in G\backslash X} |G| / |\mathrm{Stab}_G(x_\alpha)|$$

$$\boxed{\textbf{Burnside' Lemma}: \quad |G\backslash X| = \frac{1}{|G|}\sum_{g\in G} |X^g|}$$

Pf/ $F := \{(g,x) \in G\times X \mid g\cdot x = x\}$

$$G \xleftarrow{\pi_1} F \xrightarrow{\pi_2} X$$

(incidence corresp.)

(1) $|F| = \sum_{g\in G} |X^g|$

(2) $|F| = \sum_{x \in X} |\mathrm{Stab}_G(x)|$

$$\Rightarrow \sum_{g\in G} |X^g| = \sum_{x\in X} |\mathrm{Stab}_G(x)|$$

$$= \sum_{\alpha\in G\backslash X} \underbrace{\sum_{y\in\alpha} |\mathrm{Stab}_G(y)|}_{|G\cdot x_\alpha|\cdot|\mathrm{Stab}_G(x_\alpha)| = |G|} = |G|\,|G\backslash X|$$

$\square$

$\boxed{\underline{\text{Application}}: \begin{array}{l} p \text{ prime}, p>0 \\ m \in \mathbb{Z}_{\geq 1} \end{array} \Rightarrow \binom{p^r m}{p^r} \equiv m \mod (p)}$

Pf/ Use group actions! Take $G = \mathbb{Z}/p^r\mathbb{Z}$, $X = \{x_1, \dots, x_m\}$
any set with m elements.

- E = set of all p^r element subsets of $G \times X$. $\rightsquigarrow |E| = \binom{p^r m}{p^r}$
- $G \circlearrowright G \times X$ by $\sigma(g,x) = (\sigma \cdot g, x)$

So $G \circlearrowright E$ by $\sigma\{e_1, \dots, e_{p^r}\} = \{\sigma(e_1), \sigma(e_2), \dots, \sigma(e_{p^r})\}$
$\in E$ ✓

(axioms for left action are satisfied)

$E = \bigsqcup_{\alpha \in G \backslash E} \mathcal{O}_\alpha \qquad |\mathcal{O}_\alpha| \mid |G| = p^r \Rightarrow |\mathcal{O}_\alpha| = 1$ or $p \mid |\mathcal{O}_\alpha|$

$\Rightarrow |E| = \#(\text{orbits with 1 element}) \mod (p)$

Size 1 $\Rightarrow$ 2nd entry in each $p \in \alpha$ is fixed

$\Rightarrow \mathcal{O}_\alpha = \{(g, x_i) \, g \in G\} \Rightarrow m$ of them!

$\Rightarrow \binom{p^r m}{p^r} \equiv m \mod p$

□