Lecture 5: Group Actions on Sets II

Last time: G gp, X set $\rightsquigarrow$ Def left G-action on X is $G \times X \longrightarrow X$, $(g, x) \longmapsto g \cdot x$

satisfying $e \cdot x = x \quad \forall x$ & $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$

Equivalently: $G \longrightarrow \underset{\text{Set}}{\mathrm{Aut}}(X)$ is a group homomorphism

$g \longmapsto (x \longmapsto g \cdot x)$

- Orbit of $x \in X$ is $\boxed{G \cdot x} = \{ g \cdot x : g \in G \} \subseteq X$
- Stabilizer of $x \in X$ is $\boxed{\mathrm{Stab}_G x} = \{ g \in G \mid g \cdot x = x \} < G$

  (subgroup, but generally not normal)
- Fix point set for $g \in G$ is $\boxed{X^g} = \{ x \in X \mid g \cdot x = x \} \subseteq X$
- Equivalence Relation $x \underset{G}{\sim} y \iff \exists g \in G : g \cdot x = y$

  $\iff x$ & y in same G-orbit

$\rightsquigarrow \boxed{G \backslash X := X / \sim_G}$ partition of X into Equivalence Classes

Counting Lemmas

Easy observation $X = \bigsqcup_{\alpha \in G\backslash X} G \cdot x_\alpha \implies |X| = \sum_{\alpha \in G\backslash X} |G \cdot x_\alpha|.$

orbit representative

Recall $G/\mathrm{Stab}_G(x) \xrightarrow{\text{bij}} G \cdot x$ & $\mathrm{Stab}_G(x) \xrightarrow[\sim]{\text{iso}} \mathrm{Stab}_G(\sigma \cdot x)$

Corollary: (a) $|G| = |G \cdot x| \cdot |\mathrm{Stab}_G(x)| \quad \forall x \in X$

(b) $|X| = \sum_{\alpha \in G\backslash X} \dfrac{|G|}{|\mathrm{Stab}_G(x_\alpha)|}$

Burnside's Lemma (Frobenius, 1887)

$$\boxed{|G\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^g|}$$

EXAMPLES

① $S_n \circlearrowright X = \{1, \dots n\}$ Only 1 orbit $= X$

$\mathrm{Stab}_{S_n}(k) \simeq S_{n-1} \hookrightarrow S_n \Rightarrow n = |X| = \dfrac{|G|}{|\mathrm{Stab}_G(n)|} = \dfrac{n!}{(n-1)!}$

② $S_n \circlearrowright \{1, 2, \dots n\} = X$ induces $H = \langle \sigma \rangle \circlearrowright X$ for any $\sigma \in S_n$
(Eg $\sigma = (123)(4)(5)$ $n=5$)

- $|X^\sigma| =$
- Q: What is $H \backslash X$?
  A:

Eg: $n=5$ $\sigma = (123)(4)(5) \Rightarrow {}_{\langle\sigma\rangle}\backslash^{X} =$

$H = \langle \sigma \rangle \circlearrowright X = \{1, \dots, n\}$ $\quad (\sigma \in S_n)$

Burnside: $\#$ cycles in $\sigma = \left| {}_H \backslash^{X} \right| = \dfrac{1}{|H|} \sum_{g \in \langle \sigma \rangle} |X^g|$

Eg $n = 5$ $\quad \sigma = (123)(4)(5)$

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Application 1: Multinomial Coefficients

- Fix $r, n \in \mathbb{N}$ & an r-composition of $n = (a_1, \dots, a_r)$ $a_1 + \dots + a_r = n$ $a_i \in \mathbb{Z}_{>0}$

$\Rightarrow$ X = set of all partitions $P_1 \sqcup \dots \sqcup P_r$ of $\{1, \dots n\}$ with $|P_i| = a_i$.

Eg $n = 7 = 3+2+2$ $|P_1| = 3$, $|P_2| = 2$, $|P_3| = 2$

- Then $S_n \circlearrowright X$

$$\boxed{\mathrm{Stab}_{S_n}(x) = \qquad\qquad\qquad\qquad}$$

Eg: $\mathrm{Stab}_{S_n}(\{1,2,3\} \sqcup \{4,5\} \sqcup \{6,7\})$

- Consequence: Formula for Multinomial Coefficient

$$\boxed{\begin{array}{l}\#\text{ partitions of } \{1 \dots n\} \text{ with } r \text{ parts of sizes } (\lambda_1, \dots \lambda_r) \\ \qquad (|P_i| = \lambda_i \ \forall i \quad \lambda_1 + \dots + \lambda_r = n)\end{array}}$$

Application 2: Fixed pts for p-groups

Fix a prime number $p > 0$

Def: A group G is said to be a *p-group* if $|G| = p^k$ for some $k \in \mathbb{Z}_{\geq 1}$

Eg: $G = \mathbb{Z}/p^k\mathbb{Z}$ is a p-group.

Lemma: Let G be a p-group acting on a finite set X. Then

$$|X| \equiv |X^G| \mod p.$$

Here: $X^G = \bigcap_{g \in G} X^g = \{x \in X : g \cdot x = x \ \forall g \in G\}$

Proof:

Obs: This will be used in one of the Sylow Theorems.

$\boxed{\text{Application 3: } p \text{ prime}, p>0, \; m \in \mathbb{Z}_{\geq 1} \implies \binom{p^r m}{p^r} \equiv m \mod (p)}$

Pf/

$\boxed{\text{Actions of a group } G \text{ on itself}}$

$\boxed{\text{Application 4: Counting Conjugacy classes}}$

Consider $G \circlearrowright G$ by conjugation. Fix $x \in G$, Then

$\boxed{Z_G(x)}$ = centralizer of x $= Stab_G(x) = \{g \in X \mid gxg^{-1} = x\}$
$= \{g \in X \mid gx = xg\}$.

$\mathcal{C}$ = Conjugacy classes = G-orbits under conjugation.

Obs: For each $g \in G$, the set of elements of G fixed under C_g is $Z_G(g)$

By our counting lemmas:

- $|G| = \sum_{\alpha \in \mathcal{C}} |G \cdot x_\alpha| =$

- $$\underbrace{|\mathcal{C}|}_{\#\text{conj. classes}} = \frac{1}{|G|} \sum_{\alpha \in \mathcal{C}} |G^{x_\alpha}|$$