Lecture 5: Group Actions on Sets II

Last time: G gp, X set $\rightsquigarrow$ Def left G-action on X is $G \times X \longrightarrow X$, $(g, x) \longmapsto g \cdot x$

satisfying $e \cdot x = x \quad \forall x$ & $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$

Equivalently: $G \longrightarrow \mathrm{Aut}_{Set}(X)$, $g \longmapsto (x \longmapsto g \cdot x)$ is a group homomorphism

- Orbit of $x \in X$ is $\boxed{G \cdot x} = \{ g \cdot x : g \in G \} \subseteq X$
- Stabilizer of $x \in X$ is $\boxed{\mathrm{Stab}_G x} = \{ g \in G \mid g \cdot x = x \} < G$

  (subgroup, but generally not normal) [Eg: $S_n \circlearrowright \{1, \ldots n\} = X$, $\mathrm{Stab}_{S_n}(n) = S_{n-1} \ntriangleleft S_n$]
- Fix point set for $g \in G$ is $\boxed{X^g} = \{ x \in X \mid g \cdot x = x \} \subseteq X$
- Equivalence Relation $x \sim_G y \iff \exists g \in G : g \cdot x = y$

  $\iff x$ & y in same G-orbit

$\rightsquigarrow$ $\boxed{G \backslash X := X / \sim_G}$ partition of X into Equivalence Classes

$\boxed{\text{Counting Lemmas}}$

Easy observation $\quad X = \bigsqcup_{\alpha \in G\backslash X} G\cdot x_\alpha \implies |X| = \sum_{\alpha \in G\backslash X} |G\cdot x_\alpha|.$

orbit representative

Recall $\quad G/\mathrm{Stab}_G(x) \xrightarrow{\text{bij}} G\cdot x \quad \& \quad \mathrm{Stab}_G(x) \xrightarrow[\sim]{\text{iso}} \mathrm{Stab}_G(\sigma\cdot x)$

Corollary: (a) $|G| = |G\cdot x|\cdot|\mathrm{Stab}_G(x)| \quad \forall x \in X$

(b) $$|X| = \sum_{\alpha \in G\backslash X} \frac{|G|}{|\mathrm{Stab}_G(x_\alpha)|}$$

Burnside's Lemma (Frobenius, 1887)

$$\boxed{|G\backslash X| = \frac{1}{|G|}\sum_{g\in G} |X^g|}$$

Pf/ $F := \{(g,x) \in G \times X \mid g\cdot x = x\}$

$$\begin{tikzcd} & F \arrow[dl, "\pi_1"'] \arrow[dr, two heads] & \\ G & & X \end{tikzcd}$$

(incidence correspondence)

(1) $|F| = \sum_{g\in G} |X^g|$

(2) $|F| = \sum_{x \in X} |\mathrm{Stab}_G(x)|$

$$\Rightarrow \sum_{g\in G}|X^g| = \sum_{x\in X} |\mathrm{Stab}_G(x)|$$

$$= \sum_{\alpha \in G\backslash X} \underbrace{\sum_{y\in\alpha} \underbrace{|\mathrm{Stab}_G(y)|}_{=|\mathrm{Stab}_G(x_\alpha)|}}_{|G\cdot x_\alpha|\,|\mathrm{Stab}_G(x_\alpha)| = G} = |G|\,|X/G| \qquad \square$$

EXAMPLES

① $S_n \circlearrowright X = \{1, \dots n\}$ Only 1 orbit $= X$

$\text{Stab}_{S_n}(k) \simeq S_{n-1} \hookrightarrow S_n \Rightarrow n = |X| = \dfrac{|G|}{|\text{Stab}_G(n)|} = \dfrac{n!}{(n-1)!}$

② $S_n \circlearrowright \{1, 2, \dots n\} = X$ induces $H = \langle \sigma \rangle \circlearrowright X$ for any $\sigma \in S_n$

- $|X^\sigma| = \#$ 1-cycles in σ (Eg $n=5$: $X^{(123)(4)(5)} = \{4,5\}$)
- Q: What is $H \backslash X$?

  A: Write $\sigma = (i_1 \dots i_{k_1})(i_{k_1+1} \dots i_{k_2}) \dots (i_{k_s+1}, \dots i_n)$ (cycle decomp)

  Then $_H\backslash X = \{\{i_1 \dots i_{k_1}\}, \{i_{k_1+1}, \dots i_{k_2}\}, \dots \{i_{k_s+1}, \dots, i_n\}\}$

  $|_H\backslash X| = \#$ cycles in σ (including 1-cycles)

Eg: $n = 5$ $\sigma = (123)(4)(5) \Rightarrow {}_{\langle\sigma\rangle}\backslash X = \{\{1,2,3\}, \{4\}, \{5\}\}$

$\text{Stab}_{\langle\sigma\rangle}(1) = \text{Stab}_{\langle\sigma\rangle}(2) = \text{Stab}_{\langle\sigma\rangle}(3) = \{e\}$, $\text{Stab}_{\langle\sigma\rangle}(4) = \text{Stab}_{\langle\sigma\rangle}(5) = \langle\sigma\rangle$

$x_\alpha = 1, 4, 5$

$$\underset{5}{\overset{\shortparallel}{|X|}} = \sum_{\alpha \in {}_H\backslash X} \frac{|H|}{|\text{Stab}_H(x_\alpha)|} = \frac{3}{1} + \frac{3}{3} + \frac{3}{3} = 3+1+1 = 5 \checkmark$$

$H = \langle \sigma \rangle \circlearrowright X = \{1, \dots, n\}$ $\quad (\sigma \in S_n)$

Burnside: $\#$ cycles in $\sigma = |{}_H \backslash X| = \frac{1}{|H|} \sum_{g \in \langle \sigma \rangle} |X^g|$

But $\underset{i=1,\dots,o(\sigma)-1}{|X^{\sigma^i}|} = \#$ 1-cycles in $\sigma^i = \sum_{j|i} \#(\text{cycles of length } j) \cdot j$

$|H| = o(\sigma) = \text{lcm}(\text{cycle length in } \sigma)$

Conclude: $\#$ cycles in $\sigma = \frac{1}{o(\sigma)} \left(n + \sum_{i=1}^{o(\sigma)-1} \sum_{j|i} j \cdot \# \text{cycles of length } j \right)$

Eg $n = 5$ $\quad \sigma = (123)(4)(5)$

$o(\sigma) = 3 = \text{lcm}(3,1,1)$

$3 \overset{?}{=} \frac{1}{3} \left(5 + \underset{\substack{i=1\\j=1}}{(1 \cdot 2)} + (\underset{\substack{i=2\\j=1}}{1 \cdot 2} + \underset{\substack{i=2\\j=2}}{2 \cdot 0}) \right) = \frac{9}{3} = 3$ ✓

$\boxed{\text{Application 1: Multinomial Coefficients}}$

- Fix $r, n \in \mathbb{N}$ & an r-composition of $n = (a_1, \dots, a_r)$ $a_1 + \dots + a_r = n$ $a_i \in \mathbb{Z}_{>0}$
  $\Rightarrow X$ = set of all partitions $P_1 \sqcup \dots \sqcup P_r$ of $\{1, \dots n\}$ with $|P_i| = a_i$.
  Eg $n = 7 = 3+2+2$ $|P_1| = 3$, $|P_2| = 2$, $|P_3| = 2$
- Then $S_n \circlearrowright X$ & this action is transitive (one orbit!)

$$\boxed{\text{Stab}_{S_n}(x) \cong S_{a_1} \times S_{a_2} \times \dots \times S_{a_r} \text{ (with coordwise multiplication)}}$$

Eg: $\text{Stab}_{S_n}(\{1,2,3\} \sqcup \{4,5\} \sqcup \{6,7\}) \simeq S_3 \times S_2 \times S_2$

$$\Rightarrow |X| = \frac{|S_7|}{|\text{Stab}_{S_n}(x)|} = \frac{7!}{3!\,2!\,2!}$$

- Consequence: Formula for Multinomial Coefficient

$$\boxed{\begin{array}{l}\#\text{ partitions of } \{1, \dots n\} \text{ with } r \text{ parts of sizes } (\lambda_1, \dots \lambda_r) = \dfrac{n!}{\lambda_1! \cdots \lambda_r!} \\ \quad (|P_i| = \lambda_i \ \forall i \quad \lambda_1 + \dots + \lambda_r = n)\end{array}}$$

Application 2: Fixed pts for p-groups

Fix a prime number $p > 0$

Def: A group G is said to be a **p-group** if $|G| = p^k$ for some $k \in \mathbb{Z}_{\geq 1}$

Eg: $G = \mathbb{Z}/p^k\mathbb{Z}$ is a p-group.

Lemma: Let G be a p-group acting on a finite set X. Then

$$|X| \equiv |X^G| \mod p.$$

Here: $X^G = \bigcap_{g \in G} X^g = \{x \in X : g \cdot x = x \ \forall g \in G\}$

Proof: By (*) on page 1

$$|X| = \sum_{\alpha \in {}_G\backslash X} |G \cdot x_\alpha| = \underbrace{|X^G|}_{\text{orbits of size } 1} + \underbrace{\sum_{\substack{\alpha \in {}_G\backslash X \\ |G \cdot x_\alpha| > 1}} |G \cdot x_\alpha|}_{p \mid}$$

since $1 < |G \cdot x_\alpha| \mid |G| = p^k$.

$\Rightarrow |X| \equiv |X^G| \mod (p)$.

Obs: This will be used in one of the Sylow Theorems.

Application 3: p prime, $p>0$, $m \in \mathbb{Z}_{\geq 1}$ $\Rightarrow \binom{p^r m}{p^r} \equiv m \mod (p)$

Pf/ Use group actions! Take $G = \mathbb{Z}/p^r\mathbb{Z}$, $X = \{x_1, \dots, x_m\}$ any set with m elements.

- E = set of all p^r element subsets of $G \times X$. $\rightsquigarrow |E| = \binom{p^r m}{p^r}$
- $G \circlearrowright G \times X$ by $\sigma(g,x) = (\sigma \cdot g, x)$

So $G \circlearrowright E$ by $\sigma\{e_1, \dots, e_{p^r}\} = \{\sigma(e_1), \sigma(e_2), \dots, \sigma(e_{p^r})\} \in E$ ✓

(axioms for left action are satisfied)

$E = \bigsqcup_{\alpha \in G \backslash E} \mathcal{O}_\alpha$ $\quad |\mathcal{O}_\alpha| \mid |G| = p^r \Rightarrow |\mathcal{O}_\alpha| = 1$ or $p \mid |\mathcal{O}_\alpha|$

$\Rightarrow |E| \equiv \#(\text{orbits with 1 element}) \mod (p)$

Size 1 $\Rightarrow$ 2^{nd} entry in each $p \in \alpha$ is fixed

$\Rightarrow \mathcal{O}_\alpha = \{(g, x_i) \mid g \in G\}$ $\Rightarrow$ m of them!

$\Rightarrow \binom{p^r m}{p^r} \equiv m \mod p$. $\square$

Actions of a group G on itself

① Left Multiplication:

$$L : G \longrightarrow \mathrm{Aut}_{\mathrm{set}}(G)$$
$$g \longmapsto L_g : x \longmapsto g \cdot x$$

② Right Multiplication

$$R : G \longrightarrow \mathrm{Aut}_{\mathrm{set}}(G)$$
$$g \longmapsto R_g : x \longmapsto x g^{-1}$$

Obs: We need inverses to ensure $R_{g_1 g_2} = R_{g_1} \circ R_{g_2}$

$(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$ whereas $g_1 g_2 \neq g_2 g_1$ unless G is abelian.

Similarly: $G \circlearrowright X$, Y set yields $G \circlearrowright \mathrm{Fun}_{\mathrm{set}}(X, Y)$ via

$$g \cdot f : X \longrightarrow Y$$
$$x \longmapsto f(g^{-1} x)$$

- $(e \cdot f)(x) = f(x)$ ✓
- $(g_1 \cdot (g_2 \cdot f))(x) = (g_2 \cdot f)(g_1^{-1} x) = f(g_2^{-1} g_1^{-1} x)$
  $= f((g_1 g_2)^{-1} x) = (g_1 g_2 \cdot f)(x)$ ✓

③ Conjugation (HW1)

$$C : G \longrightarrow \mathrm{Aut}_{\mathrm{set}}(G)$$
$$g \longmapsto C_g$$

where $C_g(x) \stackrel{.}{=} g x g^{-1}$ $\forall x$.

gives $G \circlearrowright G$

$\boxed{\text{Application 4: Counting Conjugacy classes}}$

Consider $G \circlearrowright G$ by conjugation. Fix $x \in G$, Then:

$$\boxed{Z_G(x)} = \text{centralizer of } x = \mathrm{Stab}_G(x) = \{g \in X \mid g x g^{-1} = x\}$$
$$= \{g \in X \mid g x = x g\}.$$

$\mathcal{C}$ = Conjugacy classes = G-orbits under conjugation.

Obs: For each $g \in G$, the set of elements of G fixed under c_g is $Z_G(g)$

(Why? $h g h^{-1} = g \iff hg = gh \iff g^{-1} h g = h \iff g h g^{-1} = h$)

By our counting lemmas:

- $$|G| = \sum_{\alpha \in \mathcal{C}} |G \cdot x_\alpha| = \sum_{\alpha \in \mathcal{C}} \left| \frac{G}{Z_G(x_\alpha)} \right| = \sum_{\alpha \in \mathcal{C}} [G : Z_G(x_\alpha)].$$

- $$\underbrace{|\mathcal{C}|}_{\#\text{ conj. classes}} = \frac{1}{|G|} \sum_{\alpha \in \mathcal{C}} |G^{x_\alpha}| \underset{\text{by Obs}}{=} \underbrace{\frac{1}{|G|} \sum_{\alpha \in \mathcal{C}} |Z_G(x_\alpha)|}_{\text{average \# of elements in a centralizer}}$$

(easy case: G = abelian
$|\mathcal{C}| = |G|$
$Z_G(x_\alpha) = G$)