Lecture 8: Sylow Theorems II

Last time: Fix $p>0$ prime & set $\boxed{n = p^r m}$ with $(m:p)=1$. Fix G gp, $|G|=n$

Definition: A subgroup $P<G$ of order p^r is called a Sylow p-subgp of G

Sylow Theorems: (A) Sylow p-subgroups exist.

(B1) If $H<G$ is a p-group, then there exists a Sylow p-subgroup $P<G$ with $H \subseteq P$.

(B2) Any two Sylow p-subgroups $P, Q < G$ are conjugate to each other (ie $\exists\, g\in G$ with $Q = gPg^{-1}$)

(C) Let n_p = number of Sylow p-subgroups of G. Then (i) $n_p \equiv 1 \mod (p)$
(ii) $n_p \mid m$

TODAY: • Applications of Sylow Thms: { ① Detect simple groups ② Classification of some groups.
• Classify groups of order p^2.

Some Observations (see HW3)

Obs 1: (A) can be strengthen to arbitrary powers of p:

(A$'$) There exists subgroups H of G with $|H| = p^i$ for all $i = 0, \dots, r$.

Obs 2: original proof of Sylow(A) went through permutations & matrices / $\mathbb{F}_p$:

Obs 3: Can count n_p for $GL_n(\mathbb{F}_q)$ for any finite field $\mathbb{F}_q$ of char p $(q = p^k)$

Application 1: Simple groups

Obs: If $H \neq e, G$, $H \triangleleft G$, then G/H is group of smaller order ⇝ induction!

Def A group G is simple if it has no nontrivial proper, normal subgroups

Lemma: Assume G has a unique Sylow p-subgroup P. Then, $P \triangleleft G$.
($p \mid |G|$ & G not a p-gp)

Proof:

Proposition 1: There are no simple groups of order 28

Proposition 2: There are no simple groups of order 224.

Pf/

Proposition 3: There are no simple groups of order 56

Pf/

Classification of groups of order p^2

Lemma: If $G \neq \{e\}$ is a p-group, then its center $Z(G)$ is nontrivial

Proposition: If $|G| = p^2$, then G is abelian and $G \simeq \mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (coordwise product)

Pf/

Application: Classify groups of order 45

Fix G a finite group with $|G| = 45 = 3^2 \cdot 5$