Lecture 9: Semidirect products

Last Time:

1. Used Sylow Theorems to study non-simple groups of a fixed order

2. Classified groups of order p^2 with $p>0$ prime

TODAY. New Tool for classifying groups = Semidirect Products

3 equivalent characterizations (2 Today)

Motivating Example

- **HW1**: Saw all groups of order ≤ 5 are abelian ; S_3, D_3 are not.
- **Goal**: Classify groups G of order $6 = 2 \cdot 3$

Note: We can view the construction more generally!

(1) $G = PQ = \{g^i h^j\} = QP$

$Q \triangleleft G$, $P < G$

$P \cap Q = \{e\}$.

(2) We let P act on $Q \triangleleft G$ by conjugation:

$\alpha : P \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(Q)$ gp hom

Semidirect Products I

Definition: We say a group G is a semi-direct product of two subgroups H & N if

(i) $H \leqslant G$ & $N \trianglelefteq G$

(ii) $G = HN = \{h \cdot n \ \ h \in H, n \in N\} = NH$

(iii) $H \cap N = \{e\}$

[Write: $G = N \rtimes H$]

\rtimes

Obs: Recall 3rd Iso Theorem:

$H < G$
$N \triangleleft G$ $\rightsquigarrow$

$$\begin{tikzcd}
 & G \arrow[d, dash, "\vee"] & \\
 & HN = NH \arrow[dl, dash, "<"'] \arrow[dr, dash, "\triangleright"] & \\
H \arrow[dr, dash, "\triangleright"'] & & N \arrow[dl, dash, "<"] \\
 & H \cap N &
\end{tikzcd}$$

& $\dfrac{H}{H \cap N} \xrightarrow{\sim} \dfrac{HN}{N}$

EXAMPLES

<u>Example 1</u>: $H \triangleleft G$, then $G \simeq N \times H$ (direct product)
(coord.wise structure)

<u>Example 2</u> $G = D_n$
$= \langle s, \rho \rangle$

<u>Example 3</u>: $G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in GL_2(\mathbb{C}) \right\}$

$\boxed{\text{Semidirect Products II}}$

Fix two groups H & N and a group homomorphism

$$\boxed{\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) = \{\text{isos} : N \xrightarrow{\sim} N\}}$$

Obs: $\alpha(h_1 h_2) = \alpha(h_1) \circ \alpha(h_2)$, $\alpha(e) = id_N$, $\alpha(h^{-1}) = \alpha(h)^{-1}$

We can use this to define a binary operation of the cartesian product $N \times H$.

$$G = \{(n, h) : n \in N,\ h \in H\}$$

$$\boxed{(n_1, h_1) * (n_2, h_2) = \qquad \forall\, n_1, n_2 \in N,\ h_1, h_2 \in H.}$$

Lemma: This operation defines a group structure on G. Write $\boxed{G = N \rtimes_\alpha H}$

(H acts on N via α)

Pf/ We need to check

1. associativity,
2. neutral element exist
3. inverses exist

$\boxed{(n_1,h_1) * (n_2,h_2) = (n_1 \cdot \alpha_{(h_1)}(n_2), h_1 h_2) \ \forall\, n_1,n_2 \in N,\ h_1,h_2 \in H}$ & $\boxed{\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) \text{ gphm}}$

① **Associativity**: $g_1 = (n_1, h_1)$, $g_2 = (n_2, h_2)$, $g_3 = (n_3, h_3) \in G = N \times H$

<u>Proposition</u> I: The maps

$$H \longrightarrow G = N \rtimes_\alpha H \qquad ; \qquad N \longrightarrow G = N \rtimes_\alpha H$$
$$h \longmapsto (e_N, h) \qquad\qquad\qquad n \longmapsto (n, e_H)$$

are injective group homomorphisms. Furthermore:

(i) $H \leqslant G$, $N \trianglelefteq G$ (via the injections)

(ii) $NH = G$

(iii) $H \cap N = \{(e_N, e_H) =: e_G\}$

So $G = N \rtimes H$ (construction I)

<u>Proof</u>:

Prop 2: Given $G = N \rtimes H$, we set $\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N)$

$$h \longmapsto (n \longmapsto h n h^{-1})$$

Then, $\Phi : N \rtimes_\alpha H \longrightarrow G$ is an isomorphism of groups

$$(n, h) \longmapsto n h$$

Obs: Even though conjugation gives $N \rtimes H$, we might be able to use a different $\alpha' : H \longrightarrow \mathrm{Aut}_{Gp} N$. since different gphom α can give rise to isomorphic gps.

Summary: Semidirect Products of N & H $\longleftrightarrow$ Gp hom $\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N)$

$$N \rtimes_{\alpha} H \longleftarrow\!\!\shortmid \alpha$$

Obs: These constructions have been generalized for Hopf algebras. ("Quantum Double Constructions")

- Next Time: 3rd Characterization (involving short exact sequences)