

## Lecture 9 : Semidirect products

Last Time :

- ① Used Sylow Theorems To study  $n m$ -simple groups of a fixed order
- ② Classified groups of order  $p^2$  with  $p > 0$  prime

TODAY . New tool for classifying groups = Semidirect Products  
3 equivalent characterizations (z today)

## Motivating Example

- HW1: Saw all groups of order  $\leq 5$  are abelian ;  $S_3, D_3$  are not.
- Goal: Classify groups  $G$  of order  $6 = 2 \cdot 3$

Note: We can view the construction more generally!

①  $G = PQ = \{g^i h^\delta\} = QP$

$Q \triangleleft G$  ,  $P \leq G$

$P \cap Q = \{e\}$ .

② We let  $P$  act on  $Q \triangleleft G$  by conjugation.

$\alpha: P \longrightarrow \text{Aut}_{\text{Set}}(Q)$  st. how

## Semidirect Products I

Definition: We say a group  $G$  is a semi-direct product of two subgroups  $H \& N$  if

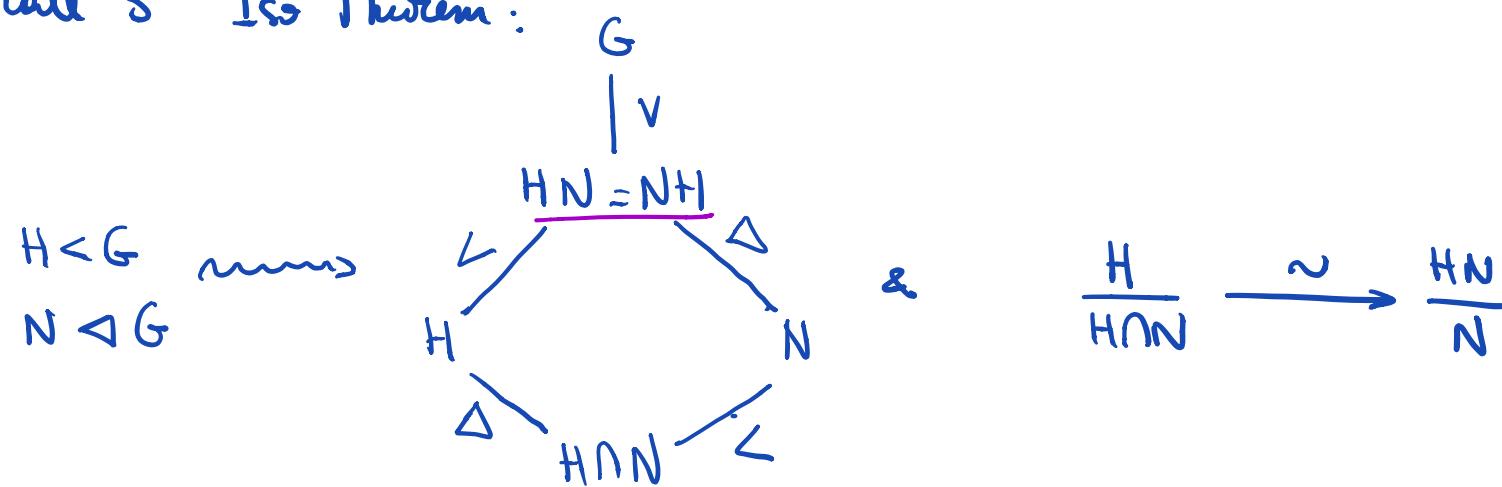
$$(i) \quad H \leq G \text{ & } N \trianglelefteq G$$

$$(ii) \quad G = HN = \{ h \cdot n \mid h \in H, n \in N \} = NH$$

$$(iii) \quad H \cap N = \{e\}$$

[Write:  $G = N \times^r H$ ]  
1<sup>r</sup> times

Obs: Recall 3<sup>rd</sup> Iso Theorem:



## EXAMPLES

Example 1:  $H \triangleleft G$ , then  $G \cong N \times H$  (direct product)  
(word-wise structure)

Example 2  $G = D_n$   
 $\langle \alpha, \beta \rangle$

Example 3:  $G = \{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in GL_2(\mathbb{C}) \}$

## Semidirect Products II

Fix two groups  $H$  &  $N$  and a group homomorphism

$$\alpha : H \longrightarrow \text{Aut}_{\text{GP}}(N) = \{ \text{isos} : N \xrightarrow{\sim} N \}$$

Obs:  $\alpha(h_1 h_2) = \alpha(h_1) \circ \alpha(h_2)$ ,  $\alpha(e) = \text{id}_N$ ,  $\alpha(h^{-1}) = \alpha(h)^{-1}$

We can use this to define a binary operation of the cartesian product  $N \times H$ .

$$G = \{(n, h) : n \in N, h \in H\}$$

$$(n_1, h_1) * (n_2, h_2) = (n_1 \alpha(h_1)(h_2), h_1 h_2)$$

Lemma: This operation defines a group structure on  $G$ . Write  $G = N \rtimes_{\alpha} H$

Pf/ We need to check  
① associativity,  
② neutral element exist  
③ inverses exist

$$(n_1, h_1) * (n_2, h_2) = (n_1 \cdot \alpha_{(h_1)}(n_2), h_1 h_2) \quad \forall n_1, n_2 \in N, h_1, h_2 \in H \quad \& \quad \alpha: H \longrightarrow \text{Aut}_{\text{Grp}}(N) \text{ gphm}$$

① Associativity:  $s_1 = (n_1, h_1)$ ,  $s_2 = (n_2, h_2)$ ,  $s_3 = (n_3, h_3) \in G = N \times H$

Proposition I: The maps  $H \longrightarrow G = N \rtimes H$ ,  $N \longrightarrow G = N \rtimes H$

$$h \longmapsto (e_N, h) \quad ; \quad n \longmapsto (n, e_H)$$

are injective group homomorphisms. Furthermore:

- (i)  $H \leq G$ ,  $N \trianglelefteq G$  (via the injections)
- (ii)  $NH = G$
- (iii)  $H \cap N = \{(e_N, e_H) =: e_G\}$

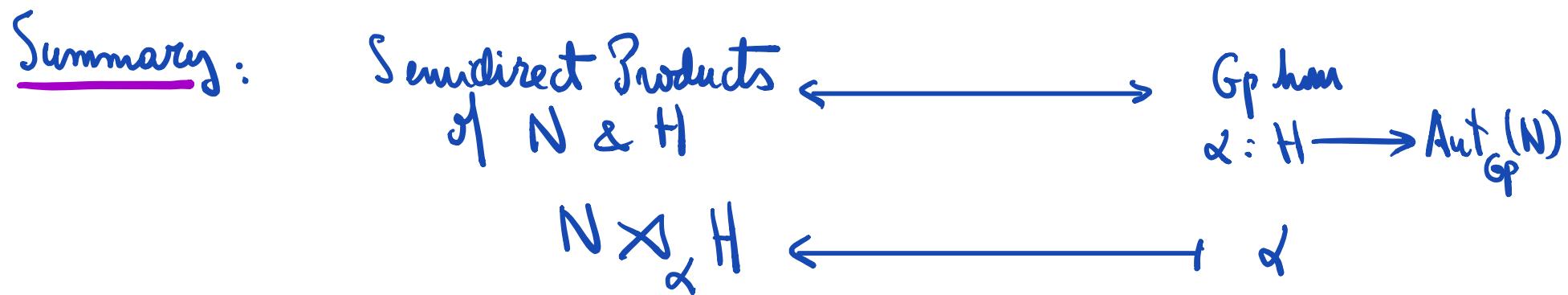
}  
So  $G = N \rtimes H$   
(construction I)

Proof:

Prop 2: Given  $G = N \rtimes H$ , we set  $\alpha: H \longrightarrow \text{Aut}_{G_P}(N)$   
 $h \longmapsto (n \mapsto hn h^{-1})$

Then,  $\Phi: N \rtimes_{\alpha} H \longrightarrow G$  is an isomorphism of groups  
 $(n, h) \longmapsto nh$

Obs: Even though conjugation gives  $N \rtimes H$ , we might be able to use a different  $\alpha': H \rightarrow \text{Aut}_{\text{Gp}} N$ . since different  $\alpha'$ s can give rise to isomorphic gps.



Obs: These constructions have been generalized for Hopf algebras.  
("Quantum Double Constructions")

• Next time: 3<sup>rd</sup> Characterization (involving short exact sequences)