Lecture 9: Semidirect products

Last Time:

1. Used Sylow Theorems to study non-simple groups of a fixed order
2. Classified groups of order p^2 with $p>0$ prime

TODAY. New Tool for classifying groups = Semidirect Products

3 equivalent characterizations (2 today)

Motivating Example

- HW1: Saw all groups of order ≤ 5 are abelian; S_3, D_3 are not.
- Goal: Classify groups G of order $6 = 2\cdot 3$

By Sylow Thm: $\left.\begin{array}{l} n_2 \equiv 1 \mod (2) \\ n_2 \mid 3 \end{array}\right\} n_2 = 1 \text{ or } 3$; $\left.\begin{array}{l} n_3 \equiv 1 \mod 3 \\ n_3 \mid 2 \end{array}\right\} \Rightarrow \boxed{n_3 = 1}$

$Syl_3(G) = \{Q\}$ with $Q = \langle h \rangle \simeq \mathbb{Z}/_{3\mathbb{Z}}$; $Syl_2(G) \ni P = \langle g \rangle \simeq \mathbb{Z}/_{2\mathbb{Z}}$

CASE 1: $n_2 = 1 \rightsquigarrow Syl_2(G) = \{P\}$

Since $P, Q \lhd G$ & $P \cap Q = \{e\}$

$\underset{HW1}{\Longrightarrow}$ P & Q mutually commute

- $\varphi : \mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{3\mathbb{Z}} \simeq P \times Q \longrightarrow G$

$(k, l) \mapsto (g^k, h^l) \longrightarrow g^k h^l$

- φ is an injective gp homomorphism
- $|G| = |P \times Q| = 6 \Rightarrow \varphi$ is iso

$$\boxed{G \simeq \mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{3\mathbb{Z}}}$$

CASE 2: $n_2 = 3 \rightsquigarrow Syl_2(G) = \{\overset{P}{\overset{\shortparallel}{P_1}}, P_2, P_3\}$

Obs: h & g cannot commute

(Otherwise $G \simeq \mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{3\mathbb{Z}}$ so $n_2 = 1$)

$G = \{1, h, h^2, g, gh, gh^2\}$ is a group

$\Rightarrow hg = gh^2$ so $ghg^{-1} = ghg = h^2$

Conclude: $\boxed{G \simeq D_3}$ where $h \longmapsto \rho$, $g \longmapsto s$

(same rlns!)

$D_3 \simeq S_3$ $\quad \rho \longmapsto (123)$, $s \longmapsto (12)$

Note: We can view the construction more generally!

1. $G = PQ = \{g^i h^j\} = QP$, $Q \triangleleft G$, $P < G$, $P \cap Q = \{e\}$. (saw in 3rd Iso Thm)

2. We let P act on $Q \triangleleft G$ by conjugation:

$$\alpha : P \longrightarrow \mathrm{Aut}_{Set}(Q) \quad \text{gp hom}$$

$$g^i \longmapsto (g^i h^j g^{-i} = \begin{cases} h^{2j} & \text{if } i=1 \\ h^j & \text{if } i=0 \end{cases}$$

$i=0$: $h^j \xrightarrow{\alpha(e)} h^j$ $\qquad$ $i=1$ $h^{j+k} \xrightarrow{\alpha(g)} h^{2(j+k)} = \alpha_{(g)}(h^j)\,\alpha_{(g)}(h^k)$

So $\boxed{\alpha : P \longrightarrow \mathrm{Aut}_{Gp}(Q) \text{ is group hom}}$

Obs: The map α provides the "commutation relation" between $g \in P$ & $h \in Q$.

Semidirect Products I

Definition: We say a group G is a **semi-direct product** of two subgroups H & N if

(i) $H \leqslant G$ & $N \trianglelefteq G$

(ii) $G = HN = \{h \cdot n \quad h \in H, n \in N\} = NH$

(iii) $H \cap N = \{e\}$

[Write: $G = N \rtimes H$] \rtimes

Obs: Recall 3^{rd} Iso Theorem:

$H < G$
$N \triangleleft G$ $\rightsquigarrow$

$$\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash, "\vee"] & \\
 & HN = NH \arrow[dl, dash, "<"'] \arrow[dr, dash, "\triangleright"] & \\
H \arrow[dr, dash, "\triangleright"'] & & N \arrow[dl, dash, "<"] \\
 & H \cap N &
\end{tikzcd}$$

(= Today)

& $\frac{H}{e} = \frac{H}{H \cap N} \xrightarrow{\sim} \frac{HN}{N} = \frac{G}{N}$

(today) (today)

Consequence: $H \simeq G/N$, so for each coset $gN \in G/N$ we can find a representative $\sigma_g \in H$ so that $\sigma_{g_1 g_2} = \sigma_{g_1} \cdot \sigma_{g_2}$.

EXAMPLES

Example 1: $H \triangleleft G$, then $G \simeq N \times H$ (direct product)
(coord.wise structure)

Example 2 $G = D_n = \langle s, \rho \rangle$ $\qquad H = \{e, s\} < G \cong \mathbb{Z}/2\mathbb{Z}$; $\quad N = \langle \rho \rangle \triangleleft G \cong \mathbb{Z}/n\mathbb{Z}$ $\qquad (s\rho^j s^{-1} = \rho^{-j})$

$$\Rightarrow D_n \simeq \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$$

Example 3: $G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in GL_2(\mathbb{C}) \right\}$ $\qquad \mathbb{C} \simeq N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\} \triangleleft G$

$(\mathbb{C}^*)^2 \simeq H = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in G \right\} < G$

- $H \cap N = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
- $G = HN = NH$ because $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \underset{H}{\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}} \underset{N}{\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}} = \underset{N}{\begin{bmatrix} 1 & b/c \\ 0 & 1 \end{bmatrix}} \underset{H}{\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}}$.

$$\Rightarrow G \simeq \mathbb{C} \rtimes (\mathbb{C}^*)^2$$

In general: Upper Δ matrices in $GL_n(\mathbb{C}) \simeq$ "$\mathbb{C}^{\frac{n(n-1)}{2}} \rtimes (\mathbb{C}^*)^n$" (parameterization, not gp iso!)

"="
[Upper Δ w/ 1's in diag]

Semidirect Products II

Fix two groups H & N and a group homomorphism

$$\boxed{\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) = \{ \text{isos} : N \xrightarrow{\sim} N \}}$$

Obs: $\alpha(h_1 h_2) = \alpha_{(h_1)} \circ \alpha_{(h_2)}$, $\alpha(e) = id_N$, $\alpha(h^{-1}) = \alpha(h)^{-1}$

We can use this to define a binary operation of the cartesian product $N \times H$.

$$G = \{ (n, h) : n \in N, h \in H \}$$

$$\boxed{(n_1, h_1) * (n_2, h_2) = (n_1 \cdot \alpha_{(h_1)}(n_2), h_1 h_2) \quad \forall n_1, n_2 \in N, h_1, h_2 \in H.}$$

Lemma: This operation defines a group structure on G. Write $\boxed{G = N \rtimes_\alpha H}$

(H acts on N via α)

Pf/ We need to check
1. associativity,
2. neutral element exist
3. inverses exist

$\boxed{(n_1,h_1) * (n_2,h_2) = (n_1 \cdot \alpha_{(h_1)}(n_2),\ h_1h_2) \quad \forall\, n_1,n_2 \in N,\ h_1,h_2 \in H}$ & $\boxed{\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) \text{ gp hom}}$

① **Associativity**: $g_1 = (n_1,h_1),\quad g_2 = (n_2,h_2),\quad g_3 = (n_3,h_3) \in G = N \times H$

$g_2 \cdot g_3 = (n_2 \cdot \alpha(h_2)(n_3),\ h_2h_3) \quad ; \quad g_1g_2 = (n_1\,\alpha_{(h_1)}(n_2),\ h_1h_2)$

$g_1(g_2g_3) = (\, n_1\ \alpha(h_1)(n_2\,\alpha_{(h_2)}(n_3))\ ,\ h_1(h_2h_3))$

α gp hom. / $\alpha(h_1)$ gp hom. $\rightarrow$

$= (n_1\ \alpha(h_1)(n_2) \cdot \alpha(h_1h_2)(n_3)\ ,\ (h_1h_2)h_3)$

$= ((n_1,h_1)\,(n_2,h_2)) \cdot (h_3, n_3) = (g_1g_2)\,g_3$

② **Neutral Element**: (e_N, e_H)

Why? $(n_1,h_1) \cdot (n_2,h_2) = (n_1,h_1)$

$(n_1\,\alpha_{(h_1)}(n_2),\ h_1h_2) = (n_1,h_1)\ \forall h_1$

forces $h_2 = e_H$ & $\alpha_{(h_1)}(n_2) = e_N\ \ \forall h_1$

Pick $h_1 = e_H \implies n_2 = e_N$.

Check: $(n_1,h_1)(e_N,e_H) = (n_1\ \underbrace{\alpha_{(h_1)}(e_N)}_{=e_N},\ h_1e_H)$

$= (n_1,h_1)$ ✓

$(e_N,e_H)(n_1,h_1) = (e_N\ \underbrace{\alpha(e_H)}_{id}(n_1),\ e_Hh_1)$

$= (n_1,h_1)$ ✓

③ **Inverses**: $(n,h)^{-1} = (\alpha(h^{-1})(n^{-1}),\ h^{-1})$

Why? $(n_1,h_1)(n_2,h_2) = (n_1\,\alpha_{(h_1)}(n_2),\ h_1h_2)$

$= (e_N, e_H)$

$\implies h_2 = h_1^{-1}$ & $\alpha(h_1)(n_2) = n_1^{-1}$

so $n_2 = \alpha(h_1)^{-1}(n_1^{-1}) = \alpha(h_1^{-1})(n_1^{-1})$

Check: $(n,h)\,(\alpha(h^{-1})(n^{-1}),\ h^{-1}) =$

$= (n\ \underbrace{\alpha_{(h)}(\alpha(h^{-1})}_{id}(n^{-1})),\ hh^{-1}) = (e_N,e_H)$ ✓

$(\alpha(h^{-1})(n^{-1}),\ h^{-1})(n,h) = (\alpha_{(h^{-1})}(n^{-1})\ \alpha(h^{-1})(n),\ h^{-1}h)$

$\alpha(h^{-1})$ gp hom. $= (e_H, e_N)$ ✓

□

Proposition: The maps $H \longrightarrow G = N \rtimes_\alpha H$; $N \longrightarrow G = N \rtimes_\alpha H$
$h \longmapsto (e_N, h)$; $n \longmapsto (n, e_H)$

are injective group homomorphisms. Furthermore:

(i) $H \leq G$, $N \trianglelefteq G$ (via the injections)

(ii) $NH = G$

(iii) $H \cap N = \{(e_N, e_H) =: e_G\}$

So $G = N \rtimes H$ (construction I)

Proof: Check the structures of H & N are compatible with that of G

$\underline{H}$: $(e_N, h_1)(e_N, h_2) = (e_N \underbrace{\alpha(h_1)(e_N)}_{= e_N}, h_1 h_2) = (e_N, h_1 h_2)$

$\Rightarrow$ $H \xrightarrow{\varphi_H} G$ is group homomorphism. Clear: $\mathrm{Ker}(\varphi_H) = \{e_H\} \Rightarrow H \hookrightarrow G$
$h \longmapsto (e_N, h)$

$\underline{N}$ $(n_1, e_H)(n_2, e_H) = (n_1 \overbrace{\alpha(e_H)}^{= id_N}(n_2), e_H e_H) = (n_1 n_2, e_H) \Rightarrow N \overset{\varphi_N}{\hookrightarrow} G$ gp hom
$n \longmapsto (n, e_H)$

• $\underline{N \triangleleft G}$ $(n, h) N (n,h)^{-1} = (n,h) N (\alpha(h^{-1})(n^{-1}), h^{-1}) \overset{?}{=} N$

$(n,h)(n_1, e_H)(\alpha(h^{-1})(n^{-1}), h^{-1}) = (n\,\alpha(h)(n_1), h)(\alpha(h^{-1})(n^{-1}), h^{-1})$

$= (n\,\alpha(h)(n_1)\,\alpha(h)(\alpha(h^{-1})(n^{-1})), h h^{-1}) = (n\,\alpha_{(h)}(n_1)\, n^{-1}, h h^{-1}) \in N$

• $\underline{NH = G}$ $(n, e_H)(e_N, h) = (n\,\alpha_{(e_H)}(e_N), e_H h) = (n, h)$ | • $\underline{N \cap H} = \{(e_N, e_H)\}$ □

Prop: Given $G = N \rtimes H$, we set $\alpha: H \longrightarrow \mathrm{Aut}_{Gp}(N)$

$$h \longmapsto (n \longmapsto h n h^{-1})$$

Then, $\Phi: N \rtimes_\alpha H \longrightarrow G$ is an isomorphism of groups

$$(n, h) \longmapsto n h$$

Proof: It is easy to check α is gp homomorphism, so $N \rtimes_\alpha H$ is well-defined.

Claim 1: Φ is a gp homomorphism.

$$\Phi((n_1, h_1)(n_2, h_2)) = \Phi((n_1 \alpha(h_1)(n_2), h_1 h_2)) = \underbrace{n_1 \alpha(h_1)(n_2)}_{\in N} \underbrace{h_1 h_2}_{\in H}$$

$$\overset{\text{def of } \alpha}{=} n_1 (h_1 n_2 h_1^{-1}) h_1 h_2 = n_1 h_1 n_2 h_2$$

$$= \Phi(n_1, h_1) \Phi(n_2, h_2).$$

Claim 2 $\mathrm{Ker}\, \Phi = \{(e_N, e_H)\}$ (because $H \cap N = \{e\}$)

Claim 3: $\mathrm{Im}\, \Phi = G$ (because $NH = G$)

By the 3 Claims, Φ is gp isomorphism $\square$

Obs: Even though conjugation gives $N \rtimes H$, we might be able to use a different $\alpha' : H \longrightarrow \mathrm{Aut}_{Gp} N$. since different gp hom α can give rise to isomorphic gps.

Summary:

$$\begin{array}{ccc} \text{Semidirect Products of } N \,\&\, H & \longleftrightarrow & \text{Gp hom } \alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) \\ N \rtimes_{\alpha} H & \longleftarrow\!\!\shortmid & \alpha \end{array}$$

(not 1-to-1 !)

can twist via

$$\begin{cases} \varphi \in \mathrm{Aut}_{Gp}(H) \\ \psi \in \mathrm{Aut}_{Gp}(N) \end{cases}$$

Obs: These constructions have been generalized for Hopf algebras. ("Quantum Double Constructions")

• **Next Time:** 3rd Characterization (involving short exact sequences)