# Lecture 10: Short exact sequences

**Last Time**: 2 characterizations of semidirect products

1. $H < G$, $N \lhd G$
   $G = NH$ $(= HN)$
   $H \cap N = \{e\}$
   $\rightsquigarrow G = N \rtimes H$

2. $\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N)$ gp hom.
   $G = N \times H$ as a set with group operation
   $(n_1, h_1) \cdot (n_2, h_2) = (n_1\, \alpha(h_1)(n_2), h_1 h_2)$
   $\rightsquigarrow G = N \rtimes_\alpha H$

1 $\Longrightarrow$ 2
$$\alpha = H \longrightarrow \mathrm{Aut}_{Gp}(N)$$
$$h \longmapsto (g \mapsto hgh^{-1})$$

2 $\Longrightarrow$ 1
$$N \hookrightarrow G \qquad , \qquad H \hookrightarrow G \qquad \text{group hom}$$
$$n \longmapsto (n, e_H) \qquad\quad h \longmapsto (e_N, h) \qquad \text{injective.}$$

**Last characterization**: via certain short exact sequences

# Short Exact Sequences

Recall (1st Isomorphism Theorem) $\varphi\, G \twoheadrightarrow G'$ gp hom, then $\frac{G}{\operatorname{Ker}\varphi} \overset{\bar{\varphi}}{\cong} G'$.

Theorem: We have an exact sequence (see definition below):

$$\mathbb{1} \longrightarrow \operatorname{Ker}(\varphi) \xrightarrow{i} G \xrightarrow{\varphi} G' \longrightarrow \mathbb{1}$$

where:

**Definition:** A sequence of group homomorphisms

$$G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3$$

is said to be exact (or exact at $G_2$) if $\operatorname{Im}\varphi = \operatorname{Ker}\Psi$.

$$\boxed{\text{ses}: \; G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3 \;\text{ with } \operatorname{Im}\varphi = \operatorname{Ker}\Psi.}$$

Examples: ① $\mathbb{1} \longrightarrow G_1 \xrightarrow{\Psi} G_2$ is exact $\Longleftrightarrow$ $\Psi$ is

② $G_1 \xrightarrow{\varphi} G_2 \longrightarrow \mathbb{1}$ is exact $\Longleftrightarrow$ $\varphi$ is

Def: An exact sequence of the form $\mathbb{1} \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3 \longrightarrow \mathbb{1}$ is usually referred to as a short exact sequence (ses). Meaning:

(i)

(ii)

1st Isomorphism Theorem

$G \xrightarrow{\varphi} G'$ gp hom $\Longrightarrow$

$$\begin{tikzcd}
\mathbb{1} \arrow[r] & \operatorname{Ker}\varphi \arrow[r] & G \arrow[r, "\varphi"] & \operatorname{Im}(\varphi) \arrow[r] & \mathbb{1} \\
\mathbb{1} \arrow[r] & \operatorname{Ker}\varphi \arrow[r] \arrow[u, "\cong\, id"] & G \arrow[r, "\pi"] \arrow[u, "\cong\, id"] & G/\operatorname{Ker}\varphi \arrow[r] \arrow[u, "\cong"', "\overline{\varphi}"] & \mathbb{1}
\end{tikzcd}$$

Obs: These 2 short exact sequences are called equivalent (vertical maps are isos, not nec. $=$).

# EXAMPLES

① (Abelian case: write trivial gp as 0)

② Pick $\det : GL_2(\mathbb{C}) \longrightarrow \mathbb{C} \setminus \{0\} =: \mathbb{C}^*$ (group under usual multiplication)

$A \longmapsto \det(A)$

is a surjective group homomorphism $\left( \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \longmapsto \lambda \right)$

$\mathrm{Ker}(\det) =$ 2×2 matrices of determinant 1 $=: SL_2(\mathbb{C})$.

$\mathbb{1} \longrightarrow SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C}) \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{1}$ is a s.e.s.

③

# $\boxed{\text{The Alternating Group}}$

Fix $\sigma \in S_n$ $\underset{\text{HW2}}{\Longrightarrow}$ $\ell(\sigma) = \#\{i<j : \sigma(i) > \sigma(j)\} =$ min # simple transpositions used to write $\sigma$.

Eg :

Set $\text{sign}: S_n \longrightarrow \{\pm 1\}$
$\sigma \longmapsto \text{sign}(\sigma) := (-1)^{\ell(\sigma)}$

Claim : sign is gp hom

Def $\boxed{A_n := \text{Ker}(\text{sign})}$ = subgroup of even permutations $(A_n \lhd S_n)$

Ex : $A_2 = \{\mathbb{1}\}$, $A_3 = \langle (123) \rangle$ , $A_4 = \langle (123), (12)(34) \rangle$

Properties : • $A_4$ is not simple $(H = \{1, (12)(34), (13)(24), (14)(23)\} \lhd A_4)$
• $A_n$ is simple for $n \geq 5$

$$\Rightarrow \boxed{\mathbb{1} \longrightarrow A_n \longrightarrow S_n \xrightarrow{\text{sign}} \{\pm 1\} \longrightarrow \mathbb{1} \quad \text{is a s.e.s.}}$$

# Sections & Retractions

Fix $1 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3 \longrightarrow 0$ ses

**Q**: Can we use $G_1 \trianglelefteq G_2$ & $G_3$ to understand/ characterize $G_2$?

**A**:

**Ex**!

**Definition:** A ses is *split* if we have a *section*, that is, a gp hom
$s : G_3 \longrightarrow G_2$ with $\Psi \circ s = id_{G_3}$.

**Definition:** A ses is *trivial* if we have a *retraction* (or projection), that is, a gp hom
$r : G_2 \longrightarrow G_1$ with $r \circ \varphi = id_{G_1}$.

$\boxed{\text{Obs 1: Not every ses splits!}}$

$\boxed{\text{Obs 2: Trivial \& split ses are different concepts!}}$

**Lemma:** A trivial ses of groups always splits

Pf/

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\varphi"] & B \arrow[r, "\Psi"] \arrow[l, bend left, dashed, "\exists r"] & C \arrow[r] \arrow[l, bend left, dashed] & 1 \end{tikzcd}$$

$A, B, C$ gps $\quad r \circ \varphi = id_A$

want: $s$ gp hom $\quad \Psi \circ s = 1_C$

$$\boxed{\text{Split \& Trivial ses will characterize } G_2 \text{ as } G_1 \rtimes_\alpha G_3 \text{ or } G_1 \times G_3}$$

**Proposition 1:** If the ses $1 \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\Psi} H \longrightarrow 1$ is Trivial, then

$$G \simeq N \times H \quad \text{(direct product)} \quad \text{when } N \xhookrightarrow{\varphi} G \;\&\; H \xhookrightarrow{s} G$$

**Proof:**

$$
\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "i"] \arrow[d, equal] & N \times H \arrow[r, "\pi_2"] \arrow[l, bend left, dashed, "\pi_1"] & H \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r, "\varphi"] & G \arrow[r, "\psi"] \arrow[u, "\iota", "\wr"'] \arrow[l, bend left, dashed, "\exists r"] & H \arrow[r] & 1
\end{tikzcd}
$$

$$
\begin{aligned}
\iota : G &\longrightarrow N \times H \\
g &\longmapsto (r(g), \psi(g))
\end{aligned}
$$

**Proposition 2:** If a ses

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "\varphi"] & G \arrow[r, "\Psi"] & H \arrow[r] \arrow[l, bend left, dashed, "s"] & 1 \end{tikzcd}$$

splits, then

$G \simeq N \rtimes H$ where $N \xhookrightarrow{\varphi} G$ & $H \xhookrightarrow{s} G$

Pf/

$$\boxed{\text{EXAMPLE}}$$

$$1 \longrightarrow A_n \xrightarrow{i} S_n \xrightarrow{\text{sign}} \{\pm 1\} \longrightarrow 1 \quad \text{splits} \implies \boxed{S_n = A_n \rtimes \mathbb{Z}/2\mathbb{Z}}$$

$$(12) \longleftarrow\!\shortmid -1$$

**Theorem:** If $N < G$ has index two, then $G \cong N \rtimes \mathbb{Z}_2$
$G$ finite