Lecture 10: Short exact sequences

Last Time: 2 characterizations of semidirect products

① $H < G$, $N \triangleleft G$

$G = NH \ (= HN)$

$H \cap N = \{e\}$

$\rightsquigarrow G = N \rtimes H$

② $\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N)$ gp hom.

$G = N \times H$ as a set with group operation

$(n_1, h_1) \cdot (n_2, h_2) = (n_1\, \alpha(h_1)(n_2),\ h_1 h_2)$

$\rightsquigarrow G = N \rtimes_\alpha H$

① $\Longrightarrow$ ②

$$\begin{aligned} \alpha : H &\longrightarrow \mathrm{Aut}_{Gp}(N) \\ h &\longmapsto (g \mapsto hgh^{-1}) \end{aligned}$$

② $\Longrightarrow$ ①

$$\begin{aligned} N &\hookrightarrow G \\ n &\longmapsto (n, e_H) \end{aligned} \quad , \quad \begin{aligned} H &\hookrightarrow G \\ h &\longmapsto (e_N, h) \end{aligned}$$

group hom injective.

Last characterization: via certain short exact sequences.

Short Exact Sequences

Recall (1st Isomorphism Theorem) $\varphi\, G \longrightarrow\!\!\!\!\rightarrow G'$ gp hom, then $\dfrac{G}{\ker\varphi} \overset{\bar\varphi}{\simeq} G'$.

Theorem: We have an exact sequence (see definition below):

$$\mathbb{1} \longrightarrow \operatorname{Ker}(\varphi) \xrightarrow{i} G \xrightarrow{\varphi} G' \longrightarrow \mathbb{1}$$

where:
- $\mathbb{1} = \{1\}$ is the trivial group
- $i : \operatorname{Ker}(\varphi) \longrightarrow G$ is the natural inclusion
- $\mathbb{1} \longrightarrow \operatorname{Ker}(\varphi)$: $1 \longrightarrow e_G$
- $G' \longrightarrow \mathbb{1}$: $g' \longrightarrow 1 \quad \forall g' \in G'$.

Definition: A sequence of group homomorphisms

$$G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3$$

is said to be exact (or exact at G_2) if $\operatorname{Im}\varphi = \operatorname{Ker}\Psi$.

Obs: $\operatorname{Ker}\Psi \triangleleft G_2$ but in general $\operatorname{Im}\varphi$ is not (unless G_2 is abelian), so this is a strong condition to impose!

$$\boxed{\underline{\text{ses}}: \; G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3 \quad \text{with } \operatorname{Im}\varphi = \operatorname{Ker}\Psi.}$$

Examples: (1) $1 \longrightarrow G_1 \xrightarrow{\Psi} G_2$ is exact $\iff$ Ψ is injective

(2) $G_1 \xrightarrow{\varphi} G_2 \longrightarrow 1$ is exact $\iff$ φ is surjective.

Def: An exact sequence of the form $1 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3 \longrightarrow 1$ is usually referred to as a short exact sequence (ses). Meaning:

(i) G_1 can be viewed as a normal subgroup of G_2 because $G_1 \xrightarrow{\sim} \operatorname{Im}\varphi < G_2$, $\operatorname{Im}\varphi = \operatorname{Ker}\Psi \triangleleft G_2$

(ii) $$G_2/\operatorname{Im}\varphi = G_2/\operatorname{Ker}\Psi \xrightarrow[\sim]{\overline{\Psi}} G_3$$ is an iso.

ses = "build G_2 out of G_1 & G_3" (G_2 is an extension of G_3 by G_1)

1st Isomorphism Theorem

$G \xrightarrow{\varphi} G'$ gp hom $\implies$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \operatorname{Ker}\varphi & \longrightarrow & G & \xrightarrow{\varphi} & \operatorname{Im}(\varphi) & \longrightarrow & 1 \\
 & & \wr\|\, id & & \wr\|\, id & & \wr\!\uparrow \overline{\varphi} & & \\
1 & \longrightarrow & \operatorname{Ker}\varphi & \longrightarrow & G & \xrightarrow{\pi} & G/\operatorname{Ker}\varphi & \longrightarrow & 1
\end{array}$$

Obs: These 2 short exact sequences are called equivalent (vertical maps are isos, not nec. =).

EXAMPLES

① (Abelian case: write trivial gp as 0)

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/5\mathbb{Z} \longrightarrow 0 \quad \text{is a s.e.s.}$$

$$m \longmapsto 5m$$

② Pick $\det : GL_2(\mathbb{C}) \longrightarrow \mathbb{C} \setminus \{0\} =: \mathbb{C}^*$ (group under usual multiplication)

$$A \longmapsto \det(A)$$

is a surjective group homomorphism ($\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \longmapsto \lambda$)

$\mathrm{Ker}(\det)$ = 2×2 matrices of determinant 1 $=: SL_2(\mathbb{C})$.

$$1 \longrightarrow SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C}) \longrightarrow \mathbb{C}^* \longrightarrow 1 \quad \text{is a s.e.s.}$$

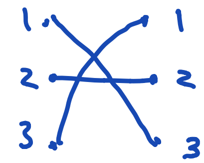
③ $$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{\varphi} D_n \xrightarrow{\Psi} \{\pm 1\} \longrightarrow 1$$

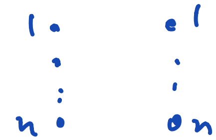
$$k \longmapsto \rho^k$$

$$s^i \rho^l \longmapsto (-1)^i$$

The Alternating Group

Fix $\sigma \in S_n$ $\overset{\text{HW2}}{\Longrightarrow}$ $\ell(\sigma) = \#\{i<j : \sigma(i) > \sigma(j)\} = \min \#$ simple transpositions used to write σ.

Eg: $\sigma = (13) = (12)(23)(12)$ so $\ell((13)) = 3 = \#$ crossings in 

<u>In general</u>: $\ell(\sigma) = \#$ crossings in string diagram  for σ.

Set $\mathrm{sign}: S_n \longrightarrow \{\pm 1\}$

$\sigma \longmapsto \mathrm{sign}(\sigma) := (-1)^{\ell(\sigma)}$

<u>Claim</u>: sign is gp hom (Proof by picture; $\ell(\sigma\tau) \equiv \ell(\sigma) + \ell(\tau) \pmod 2$)

<u>Def</u> $\boxed{A_n := \mathrm{Ker}(\mathrm{sign})}$ = subgroup of even permutations $(A_n \triangleleft S_n)$

Ex: $A_2 = \{\mathbb{1}\}$, $A_3 = \langle (123) \rangle$, $A_4 = \langle (123), (12)(34) \rangle$

<u>Properties</u>:
- A_4 is not simple $(H = \{1, (12)(34), (13)(24), (14)(23)\} \triangleleft A_4)$
- A_n is simple for $n \geq 5$

$$\Rightarrow \boxed{\mathbb{1} \longrightarrow A_n \longrightarrow S_n \xrightarrow{\text{sign}} \{\pm 1\} \longrightarrow \mathbb{1} \text{ is a s.e.s.}}$$

Sections & Retractions

Fix $\mathbb{1} \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3 \longrightarrow 0$ ses

$\exists r?$ $\exists s?$

Q: Can we use $G_1 \triangleleft G_2$ & G_3 to understand/ characterize G_2 ?

A: Usually knowing $N \triangleleft G$ & G/N does not characterize G !

Ex: $\mathbb{1} \longrightarrow \langle \rho^2 \rangle \longrightarrow D_4 \longrightarrow D_4/\langle \rho^2 \rangle \longrightarrow \mathbb{1}$

$\langle \rho^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$, $\quad D_4/\langle \rho^2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ $\;(\langle \rho \rangle \times \langle s \rangle)$

$\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$, $\quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong Q_8/\{\pm 1\}$ $\;(\langle i \rangle \times \langle j \rangle)$

$\mathbb{1} \longrightarrow \{\pm 1\} \longrightarrow Q_8 \longrightarrow Q_8/\{\pm 1\} \longrightarrow \mathbb{1}$.

order $4 = 2^2$ & non cyclic.

But $D_4 \not\cong Q_8$

Conclude: Answer will depend on extra properties of φ and/or Ψ !

Definition: A ses is <u>split</u> if we have a <u>section</u>, that is, a gp hom $s: G_3 \longrightarrow G_2$ with $\Psi \circ s = id_{G_3}$ ($\Rightarrow s$ is injective!)

Definition: A ses is <u>trivial</u> if we have a <u>retraction</u> (or projection), that is, a gp hom $r: G_2 \longrightarrow G_1$ with $r \circ \varphi = id_{G_1}$. ($\Rightarrow r$ is surjective!)

$$\boxed{\text{Obs 1: Not every ses splits!}}$$

Ex:
$$\mathbb{1} \longrightarrow \{\pm 1\} \longrightarrow Q_8 \longrightarrow Q_8/\{\pm 1\} \longrightarrow \mathbb{1}. \quad \text{ses}$$

$\nexists s$

$Q_8/\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Q_8: only elements of order $2 = \pm 1$

$Q_8/\{\pm 1\}$: <u>all</u> elements have order 2 (except $\{\pm 1\}$)

$$\boxed{\text{Obs 2: Trivial \& split ses are different concepts!}}$$

Ex:
$$\mathbb{1} \longrightarrow A_3 \xrightarrow{i} S_3 \xrightarrow{\text{sign}} \{\pm 1\} \longrightarrow \mathbb{1} \qquad s(-1) = (12).$$

$\nexists r$ gp hom $\qquad \exists s$ gp hom

Why? Any $r: S_3 \twoheadrightarrow A_3$ sends (ij) to 1 ($o(ij) = 2$ & $r(ij) \mid 3$) $\Rightarrow r: S_3 \longrightarrow \mathbb{1}$.

Lemma: A trivial ses of groups always splits

Pf/

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\varphi"] & B \arrow[r, "\Psi"] \arrow[l, bend left, dashed, "\exists r"] & C \arrow[r] \arrow[l, bend left, dashed] & 1 \end{tikzcd}$$

A, B, C gps $\quad r \circ \varphi = id_A$

want: s gp hom $\quad \Psi \circ s = 1_C$

Consider $\boxed{\Psi_{|\ker r} : \ker r \longrightarrow C}$ gp hom

Claim 1: $\Psi_{|\ker r}$ is injective.

Pf/ Pick $b \in \ker r \cap \ker \Psi$.

$\Rightarrow b \in \ker \Psi = \operatorname{Im} \varphi$ ← exactness

so $b = \varphi(a)$ for $a \in A$.

Then $e_A = r(b) = \underbrace{r \circ \varphi}_{1_A}(a) = a$

$\Rightarrow b = \varphi(e_A) = e_B$

Claim 2: $\Psi_{|\ker r}$ is surjective

For $c \in C$, pick $b \in B$ with $\Psi(b) = c$

Other choices? ALL $b' = b\varphi(a)$ with $a \in A$

Set $b' = b\, \varphi \circ \underbrace{r(b^{-1})}_{\in A} \in \ker r$ since

$r(b') = r(b) \underbrace{r \circ \varphi}_{= 1_A} \circ r(b^{-1}) = e_B$

$\Psi(b') = \Psi(b) = c \Longrightarrow c \in \operatorname{Im} \Psi_{|\ker r}$

Then $\exists\ s : C \longrightarrow \ker r < B$ gp hom with $\Psi \circ s = 1_C$. $\square$

Split & Trivial ses will characterize G_2 as $G_1 \rtimes_\alpha G_3$ or $G_1 \times G_3$

Proposition 1: If the ses $1 \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\Psi} H \longrightarrow 1$ is Trivial, then $G \simeq N \times H$ (direct product) when $N \overset{\varphi}{\hookrightarrow} G$ & $H \overset{s}{\hookrightarrow} G$

Proof: Assume $\exists\, r : G \longrightarrow N$ retraction. Then:

$$
\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "i"] \arrow[d, equal] & N \times H \arrow[r, "\pi_2"] \arrow[l, dashed, bend left, "\pi_1"] & H \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r, "\varphi"] & G \arrow[r, "\psi"] \arrow[u, "\eta", "\wr"'] \arrow[l, dashed, bend left, "\exists r"] & H \arrow[r] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N \times H \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
N & & H
\end{tikzcd}
$$

Define $\eta : G \longrightarrow N \times H$ via $\eta(g) = (r(g), \psi(g))$ gp hom ✓

Claim 1: η is injective.

Pf/ If $\eta(g) = (e_N, e_H)$ then $\Psi(g) = e_H$, so $g \in \operatorname{Ker} \Psi = \operatorname{Im} \varphi$.

$$
\left.\begin{array}{l}
\text{Then, } \exists\, x \in N \text{ with } g = \varphi(x) \\
\Rightarrow e_N = r(g) = r \circ \varphi(x) = x
\end{array}\right\} \Rightarrow g = \varphi(e_N) = e_G .
$$

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "i"] \arrow[d, equal] & N \times H \arrow[r, "\pi_2"] \arrow[l, dashed, bend left, "\pi_1"] & H \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r, "\varphi"] & G \arrow[r, "\psi"] \arrow[u, "\eta"] \arrow[l, dashed, bend left, "\exists r"] & H \arrow[r] & 1
\end{tikzcd}$$

$$\eta : G \longrightarrow N \times H$$
$$g \longmapsto (r(g), \psi(g))$$

<u>Claim 2</u>: η is surjective.

Pf/ Pick $x \in N$ & $h \in H$. Choose $g \in G$ with $\psi(g) = h$ ($\exists$ because ψ surj)

Take $\boxed{\tilde{g} = g\,(\varphi \circ r(g))^{-1}\,\varphi(x) \in G}$

$$\Rightarrow \psi(\tilde{g}) = \psi(g)\,\psi(\varphi \circ r(g^{-1}))\,\underbrace{\psi \circ \varphi(x)}_{=e_H} = \underbrace{\psi(g)\,\psi \circ \varphi(\underbrace{r(g^{-1})}_{\in N})}_{=e_H} = \psi(g) = h$$

$$r(\tilde{g}) = r(g)\,\underbrace{r}_{id_N}(\varphi \circ r(g^{-1}))\,\underbrace{r \circ \varphi}_{id_N}(x) = \underbrace{r(g)\,r(g^{-1})}_{=e_G} \cdot x = x \in N$$

So $\eta(\tilde{g}) = (x, h)$ $\square$

<u>Obs</u>: It is easy to check all squares commute.

Proposition 2: If a ses $1 \to N \xrightarrow{\varphi} G \underset{s}{\overset{\Psi}{\rightleftarrows}} H \to 1$ splits, then $G \simeq N \rtimes H$ where $N \overset{\varphi}{\hookrightarrow} G$ & $H \overset{s}{\hookrightarrow} G$

Pf/ Use def.! $N \underset{\varphi}{\trianglelefteq} G$ & $H \underset{s}{\leq} G$.

Claim 1: $s(H) \cap \varphi(N) = \{e\}$

Pf/ Pick $g \in s(H) \cap \varphi(N)$

$\Rightarrow g = s(h) = \varphi(x) \quad x \in N,\ h \in H$

$\Rightarrow \Psi(g) = \Psi \circ s(h) = h$

$= \Psi \circ \varphi(x) = e_H$

$\Rightarrow \boxed{g = s(e_H) = e_G}$

Claim 2: $NH = \{\varphi(x)\, s(h) : x \in N, h \in H\} = G$

Pf/ Pick $g \in G \Rightarrow \Psi(g) \in H$

Pick $\boxed{\tilde{g} = s \circ \Psi(g)}$ $\Rightarrow \Psi(g) = \Psi(\tilde{g})$

$\Rightarrow \tilde{g}^{-1} g \in \operatorname{Ker} \Psi = \operatorname{Im} \varphi \Rightarrow \tilde{g}^{-1} g = \varphi(x) \quad x \in N$

exactness

$\Rightarrow g = \tilde{g}\, \varphi(x) = s \circ \Psi(g)\, \varphi(x)$

$= \underbrace{s\circ\Psi(g)\, \overset{\Uparrow}{\varphi(x)}\, (s\circ\Psi(g))^{-1}}_{\in N}\ \underbrace{s \circ \Psi(g)}_{\in H} \in NH$

$N \trianglelefteq G$

$\Rightarrow G = \varphi(N) \rtimes \varphi(H) \simeq N \rtimes H$

Obs: $\alpha : H \longrightarrow \operatorname{Aut}_{Gp}(N) \qquad h \longmapsto (n \longmapsto s(h)\varphi(n)s(h^{-1}))$

EXAMPLE

$$1 \longrightarrow A_n \xrightarrow{i} S_n \xrightarrow{\text{sign}} \{\pm 1\} \longrightarrow 1 \text{ splits} \Rightarrow \boxed{S_n = A_n \rtimes \mathbb{Z}/2\mathbb{Z}}$$

$$(12) \longleftarrow\!\shortmid -1$$

Theorem: If $N < G$ has index two, then $G \cong N \rtimes \mathbb{Z}_2$
G finite

Pf/. $N \triangleleft G$ because $G/N = \{eN, gN\}$ for some $g \notin N$
$N\backslash G = \{eN, Ng\}$

so $gN = Ng$.

- Pick $h \in G \setminus N$ of order 2 ($g^2 \in N$ so $g^2 = e$ or $o(g^2) = k$, $k > 1$, so $o(g^k) = 2$)

Then

$$\left.\begin{array}{ll} H = \langle h \rangle < G & \\ H \cap N = \{e\} & \text{since } h \notin N \\ NH = G & \text{because } |N\backslash G| = 2 \end{array}\right\} G = N \rtimes H \simeq N \rtimes \mathbb{Z}_2$$

$\square$