Lecture 11: Composition Series & Zassenhaus Lemma

Recall: A group S is called simple if $\{e\}$ & S are the only normal subgroups of S

TODAY: Study groups by chain of subgroups with normality properties

Def: A composition series of a group G is a finite sequence of subgroups of G

$$\boxed{\Sigma: \qquad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = \{e\}}$$

such that $G_{j+1} \triangleleft G_j$ (normal) for all $j = 0, \dots, k-1$.

The successive quotients: $gr_i(G) := G_i / G_{i+1}$ $\quad 0 \leq i \leq k-1$. (graded pieces)

(Other notation: $gr_i^{\Sigma}(G)$ if Σ is not clear from context.)

$$\boxed{\Sigma: \quad G = G_0 \rhd G_1 \rhd \cdots \rhd G_k = \{e\}}$$

Def: A composition series Σ' is said to refine Σ (or be finer than Σ) if Σ is obtained from Σ' by omitting some terms.

More precisely: $\Sigma': \; G = G'_0 \supseteq \cdots \supseteq G'_m = \{e\}$

$$\Sigma: \; G = G_0 \supseteq \cdots \supseteq G_n = \{e\}$$

Σ' is finer than Σ if $n \leq m$ and there exists an order-preserving injective map $\Phi: \{0, \ldots, n\} \longrightarrow \{0, \ldots, m\}$ with $G_j = G'_{\phi(j)}$ $\forall j$.

⚠ In general, omitting some terms of a comp series is <u>NOT</u> a comp series

<u>Why?</u> for $j > i+1$ G_j is <u>not</u> normal in G_i.

<u>EXAMPLE:</u>

Equivalence for composition series

Fix two composition series of two groups G & H:

$$\Sigma_1 : \quad G = G_0 \supseteq \dots \supseteq G_m = \{e\}$$

$$\Sigma_2 : \quad H = H_0 \supseteq \dots \supseteq H_n = \{e\}$$

Def: We say Σ_1 & Σ_2 are equivalent if

(i)

(ii)

Common refinements up to equivalence

Theorem (Schreier) Let Σ_1 & Σ_2 be two composition series of a group G. Then, there exist composition series Σ_1' & Σ_2' finer than Σ_1 & Σ_2, respectively such that Σ_1' & Σ_2' are equivalent.

[Interpretation : any two composition series have a "common refinement", up to equivalence]

$\boxed{\text{Proof of Schreier's Thm}}$

(Equivalent common refinements for Σ_1 & Σ_2)

Write $\boxed{\Sigma_1 : G = H_0 \geq \cdots \geq H_n = \{e\} \; ; \quad \Sigma_2 : G = K_0 \geq \cdots \geq K_p = \{e\}}$

$$\boxed{\Sigma_1 : G = H_0 \geq \cdots \geq H_n = \{e\} \; ; \quad \Sigma_2 : G = K_0 \geq \cdots \geq K_p = \{e\}}$$

$$\boxed{H'_{i,j} := H_{i+1}(H_i \cap K_j)} \quad \& \quad \boxed{K'_{j,i} := K_{j+1}(H_i \cap K_j)} \qquad \begin{matrix} i = 0, \dots, n-1, \\ j = 0, \dots, p-1 \end{matrix}$$

<u>Claim</u>: (i) $H'_{i,j+1} \lhd H'_{i,j} \quad , \quad K'_{j,i+1} \lhd K'_{j,i}$

(ii) $H'_{i,j} / H'_{i,j+1} \simeq K'_{j,i} / K'_{j,i+1}$

To simplify notation, write $H = H_i \rhd H' = H_{i+1}$

$K = K_j \rhd K' = K_{j+1}$

The claims will follow using Zassenhaus' Lemma. $\square$

<u>Lemma (Zassenhaus)</u>: Fix a group G, H, K two subgroups of G & $H' \lhd H$, $K' \lhd K$. Then:

(i) $H' \cdot (H \cap K') \lhd H' \cdot (H \cap K)$

$K' \cdot (H' \cap K) \lhd K' \cdot (H \cap K)$

(ii) $\dfrac{H' \cdot (H \cap K)}{H' \cdot (H \cap K')} \simeq \dfrac{K' \cdot (H \cap K)}{K' \cdot (H' \cap K)}$

Proof of Zassenhaus Lemma

Guiding Picture (statement)

H, K, H', K' $\quad H' \triangleleft H \quad K' \triangleleft K$

STEP 1 $\boxed{(H' \cap K) \cdot (K' \cap H) \triangleleft H \cap K}$

Guiding Picture (statement)

H, K, H', K' $\quad H' \triangleleft H \quad K' \triangleleft K$

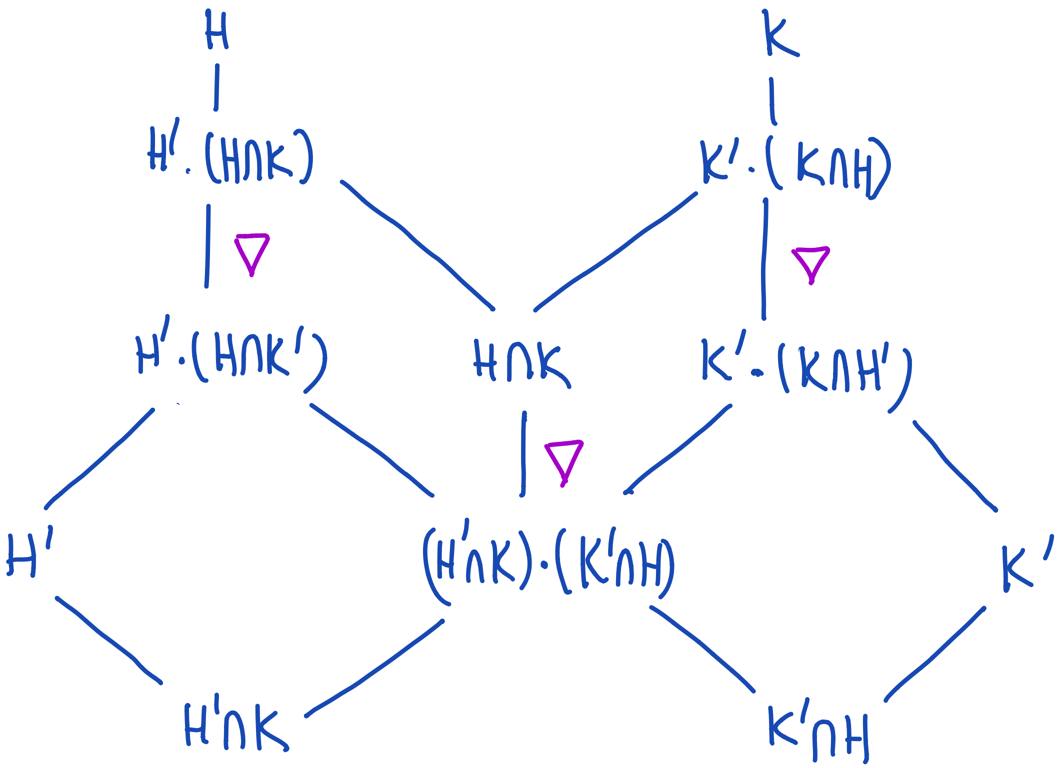
$$\begin{tikzcd}[column sep=small]
& H \arrow[d, dash] & & & & K \arrow[d, dash] & \\
& H'\cdot(H\cap K) \arrow[d, dash, "\triangledown"] \arrow[drr, dash] & & & & K'\cdot(K\cap H) \arrow[d, dash, "\triangledown"] \arrow[dll, dash] & \\
& H'\cdot(H\cap K') \arrow[dl, dash] \arrow[drr, dash] & & H\cap K \arrow[d, dash, "\triangledown"] & & K'\cdot(K\cap H') \arrow[dll, dash] \arrow[dr, dash] & \\
H' \arrow[dr, dash] & & & (H'\cap K)\cdot(K'\cap H) \arrow[dll, dash] \arrow[drr, dash] & & & K' \arrow[dl, dash] \\
& H'\cap K & & & & K'\cap H &
\end{tikzcd}$$

$$\frac{H'\cdot(H\cap K)}{H'\cdot(H\cap K')} \simeq \frac{K'\cdot(H\cap K)}{K'\cdot(H'\cap K)} \simeq \frac{H\cap K}{(H'\cap K)(K'\cap H)}$$

STEP 2 $\boxed{H'(H\cap K') \triangleleft H'(H\cap K)}$

Guiding Picture (statement)

H, K, H', K' $\quad H' \triangleleft H \quad K' \triangleleft K$



$$\frac{H'\cdot(H\cap K)}{H'\cdot(H\cap K')} \simeq \frac{K'\cdot(H\cap K)}{K'\cdot(H'\cap K)} \simeq \frac{H\cap K}{(H'\cap K)(K'\cap H)}$$

STEP 3

Claim: $(H \cap K) \cap (H' \cdot (H \cap K')) = (H' \cap K) \cdot (K' \cap H)$ if $\begin{array}{l} H' \triangleleft H \\ K' \triangleleft K \end{array}$