Lecture 11: Composition Series & Zassenhaus Lemma

Recall: A group S is called simple if $\{e\}$ & S are the only normal subgroups of S

Examples: A_n $n \geq 5$ are simple (next week)

$\mathbb{Z}/p\mathbb{Z}$ $p>0$ prime & $PSL_n := SL_n / Z(SL_n)$ are simple

TODAY: Study groups by chain of subgroups with normality properties

Def: A composition series of a group G is a finite sequence of subgroups of G

$$\boxed{\Sigma : \quad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = \{e\}}$$

such that $G_{j+1} \lhd G_j$ (normal) for all $j = 0, \dots, k-1$.

The successive quotients : $gr_i(G) := G_i / G_{i+1}$ $0 \leq i \leq k-1$. (graded pieces)

(Other notation : $gr_i^{\Sigma}(G)$ if Σ is not clear from context.)

$$\boxed{\Sigma: \quad G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_k = \{e\}}$$

Ex 1: $\Sigma_0 : \mathbb{Z}/6\mathbb{Z} \supseteq \{e\}$ is a composition series of $\mathbb{Z}/6\mathbb{Z}$.

Def.: A composition series Σ' is said to refine Σ (or be finer than Σ) if Σ is obtained from Σ' by omitting some terms.

More precisely: $\Sigma' : G = G'_0 \supseteq \cdots \supseteq G'_m = \{e\}$

$\Sigma : G = G_0 \supseteq \cdots \supseteq G_n = \{e\}$

Σ' is finer than Σ if $n \leq m$ and there exists an order-preserving injective map $\Phi : \{0, \ldots, n\} \longrightarrow \{0, \ldots, m\}$ with $G_j = G'_{\phi(j)}$ $\forall j$.

Ex 2 $\Sigma_1 : G = \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/3\mathbb{Z} \supseteq \{e\}$ no refinement

$\Sigma_2 : G = \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$ ——

$gr_0^{\Sigma_1}(G) = \mathbb{Z}/6\mathbb{Z} / \mathbb{Z}/3\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} = gr_1^{\Sigma_2}(G)$, $gr_1^{\Sigma_1} = \mathbb{Z}/3\mathbb{Z} = gr_0^{\Sigma_2}(G)$.

Both Σ_1 & Σ_2 refine Σ_0 via $\phi : \{0,1\} \longrightarrow \{0 1 2\}$ $\phi(0) = 0$ $\phi(1) = 2$

⚠ In general, omitting some terms of a comp series is **NOT** a comp series

Why? for $j > i+1$, G_j is not normal in G_i.

EXAMPLE: $G = D_4 \supseteq \langle \rho^2, s \rangle \supseteq \langle s \rangle \supseteq \{e\}$ $\quad \left(\begin{array}{l} s\rho^{2i} = \rho^{-2i} s \\ s^{-1} = s,\ \rho^4 = 1 \end{array} \right)$

$$\begin{cases} G_1 = \langle \rho^2, s \rangle \lhd G & \left(\begin{array}{ll} \rho\rho^2\rho^{-1} = \rho^2 \in G_1, & \rho s \rho^{-1} = \rho^2 s \in G_1 \\ s\rho^2 s^{-1} = \rho^2 \in G_1, & s s s^{-1} = s \in G_1 \end{array} \right) \\ G_2 = \langle s \rangle \lhd \langle \rho^2, s \rangle & \left(\begin{array}{l} \rho^2 s \rho^{-2} = \rho^2 \rho^2 s = \rho^4 s = s \in G_1 \\ s s s^{-1} = s \in G_1 \end{array} \right) \\ G_3 = \{e\} \lhd \langle s \rangle & \end{cases}$$

$$gr_0(D_4) = \frac{\langle \rho, s \rangle}{\langle \rho^2, s \rangle} \cong \mathbb{Z}/2\mathbb{Z}, \quad gr_2(D_4) = \frac{\langle s \rangle}{\langle e \rangle} \cong \mathbb{Z}/2\mathbb{Z}$$

$$gr_1(D_4) = \frac{\langle \rho^2, s \rangle}{\langle s \rangle} \cong \langle \rho^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

- We can't omit $\langle \rho^2 s \rangle$ and have a composition series because $\langle s \rangle \ntriangleleft D_4$.
- We can omit $\langle s \rangle \rightsquigarrow \Sigma_2 : D_4 \supseteq \langle \rho^2, s \rangle \supseteq \{e\}$ $\quad gr_0^{\Sigma_2}(D_4) \cong \mathbb{Z}/2\mathbb{Z}$, $gr_1^{\Sigma_2}(D_4) \cong D_2$

Equivalence for composition series

Fix two composition series of two groups G & H:

$$\Sigma_1 : \quad G = G_0 \supseteq \dots \supseteq G_m = \{e\}$$

$$\Sigma_2 : \quad H = H_0 \supseteq \dots \supseteq H_n = \{e\}$$

<u>Def</u>: We say Σ_1 & Σ_2 are <u>equivalent</u> if

(i) $m = n$

(ii) $\exists\ \sigma \in S_n = \mathrm{Aut}_{Set}(\{0, \dots, n-1\})$ such that

$$\boxed{gr_i^{\Sigma_1}(G) = gr_{\sigma(i)}^{\Sigma_2}(H) \quad \forall i}$$

<u>Ex</u>: $G = \mathbb{Z}/4\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$

are equivalent $(\sigma = id)$

$H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$

Common refinements up to equivalence

Theorem (Schreier) Let Σ_1 & Σ_2 be two composition series of a group G. Then, there exist composition series Σ_1' & Σ_2' finer than Σ_1 & Σ_2, respectively such that Σ_1' & Σ_2' are equivalent.

[Interpretation: any two composition series have a "common refinement", up to equivalence]

Ex: $G = \mathbb{Z}/6\mathbb{Z}$ $\qquad \Sigma_1 = \Sigma_1' : \mathbb{Z}/6\mathbb{Z} \geq \mathbb{Z}/3\mathbb{Z} \geq \{e\}$

$\Sigma_2 = \Sigma_2' : \mathbb{Z}/6\mathbb{Z} \geq \mathbb{Z}/2\mathbb{Z} \geq \{e\}$

- graded pieces: $\mathbb{Z}/2\mathbb{Z}$ & $\mathbb{Z}/3\mathbb{Z}$

$$gr_0^{\Sigma_1} = gr_1^{\Sigma_2} = \mathbb{Z}/3\mathbb{Z}$$
$$gr_1^{\Sigma_1} = gr_0^{\Sigma_2} = \mathbb{Z}/2\mathbb{Z}$$

- Σ_1 & Σ_2 are equivalent via $\sigma \in S_2$: $\begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}$

$\boxed{\text{Proof of Schreier's Thm}}$

(Equivalent common refinements for Σ_1 & Σ_2)

Write $\boxed{\Sigma_1 : G = H_0 \geq \cdots \geq H_n = \{e\} \; ; \quad \Sigma_2 : G = K_0 \geq \cdots \geq K_p = \{e\}}$

Idea: ① For each $i = 0, \ldots, n-1$, use Σ_2 to insert $(p-1)$ many groups $\{H'_{ij}\}_{j=1}^{p-1}$ in between H_i & H_{i+1} $\rightsquigarrow$ get Σ'_1 finer than Σ_1

② Similarly, use Σ_1 to insert $(n-1)$ many subgroups $\{K'_{j,i}\}_{i=1}^{n-1}$ in between K_j & K_{j+1} $\rightsquigarrow$ get Σ'_2 finer than Σ_2.

③ Show Σ'_1 & Σ'_2 are equivalent

Define: $\boxed{H'_{i,j} := H_{i+1}(H_i \cap K_j)}$ & $\boxed{K'_{j,i} := K_{j+1}(H_i \cap K_j)}$ $\quad i = 0, \ldots, n-1, \; j = 0, \ldots, p-1$

- It is clear that: $H'_{i,0} = H_i$, $H'_{i,p} = H_{i+1}$; $K'_{j,0} = K_j$, $K'_{j,n} = K_{j+1}$.
  - $H'_{i,j+1} < H'_{i,j}$; $K'_{j,i+1} < K'_{j,i}$ $\forall i,j$ are subgroups by 3rd Iso theorem ($H < G$, $N \triangleleft G \Rightarrow N \cdot H = HN < G$)
- Missing: normal condition & identification of graded pieces!

$\boxed{\Sigma_1: G = H_0 \geq \cdots \geq H_n = \{e\} \; ; \quad \Sigma_2: G = K_0 \geq \cdots \geq K_p = \{e\}}$

$\boxed{H'_{i,j} := H_{i+1}(H_i \cap K_j)}$ & $\boxed{K'_{j,i} := K_{j+1}(H_i \cap K_j)}$ $\quad$ $i = 0, \dots, n-1,$ $j = 0, \dots, p-1$

<u>Claim</u>: (i) $H'_{i,j+1} \lhd H'_{i,j}$, $K'_{j,i+1} \lhd K'_{j,i}$

(ii) $H'_{i,j} / H'_{i,j+1} \simeq K'_{j,i} / K'_{j,i+1}$

To simplify notation, write $H = H_i \rhd H' = H_{i+1}$

$K = K_j \rhd K' = K_{j+1}$

The claims will follow using Zassenhaus' Lemma. $\square$

<u>Lemma (Zassenhaus)</u>: Fix a group G, H, K two subgroups of G & $H' \lhd H$, $K' \lhd K$. Then:

(i) $H' \cdot (H \cap K') \lhd H' \cdot (H \cap K)$

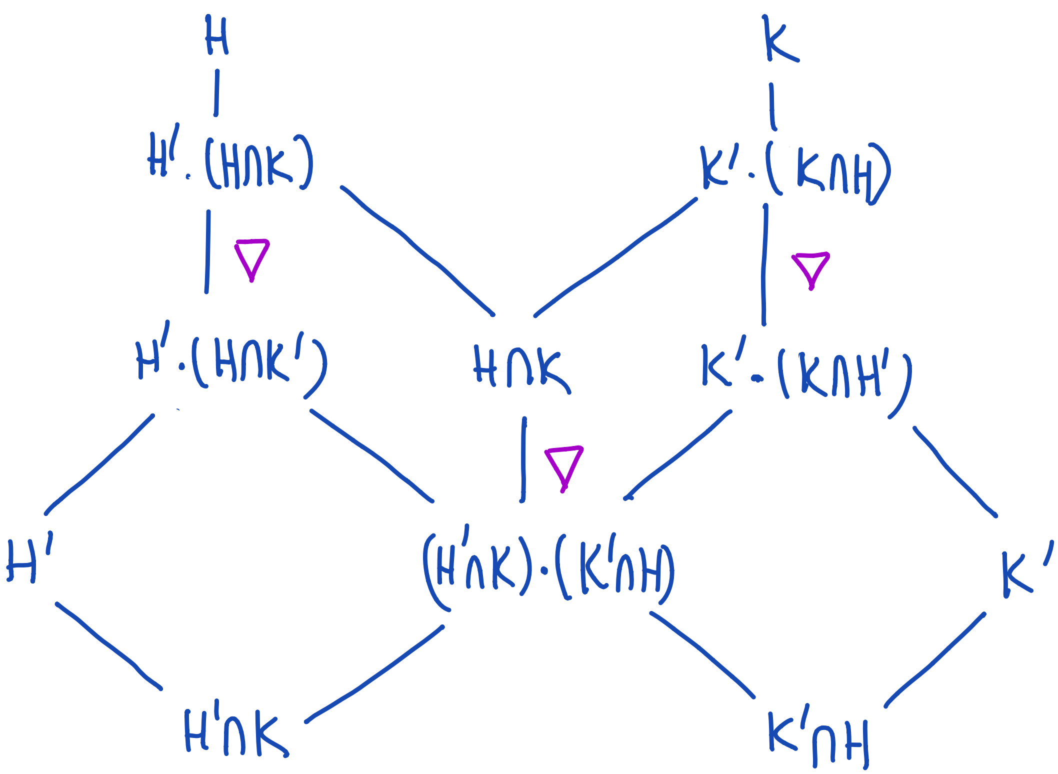
$K' \cdot (H' \cap K) \lhd K' \cdot (H \cap K)$

(ii) $\dfrac{H' \cdot (H \cap K)}{H' \cdot (H \cap K')} \simeq \dfrac{K' \cdot (H \cap K)}{K' \cdot (H' \cap K)}$

Proof of Zassenhaus Lemma

Guiding Picture (statement)

H, K, H', K' $\quad H' \lhd H \quad K' \lhd K$



$$\frac{H'\cdot(H\cap K)}{H'\cdot(H\cap K')} \simeq \frac{K'\cdot(H\cap K)}{K'\cdot(H'\cap K)} \simeq \frac{H\cap K}{(H'\cap K)(K'\cap H)}$$

STEP ① $\boxed{(H'\cap K)\cdot(K'\cap H) \lhd H\cap K}$

Pf/ By 3rd Iso Thm :

- $H' \lhd H$ so $H'\cap K \lhd H\cap K$
- $K' \lhd K$ so $K'\cap H \lhd H\cap K$

$\Rightarrow (H'\cap K)(K'\cap H) < H\cap K$

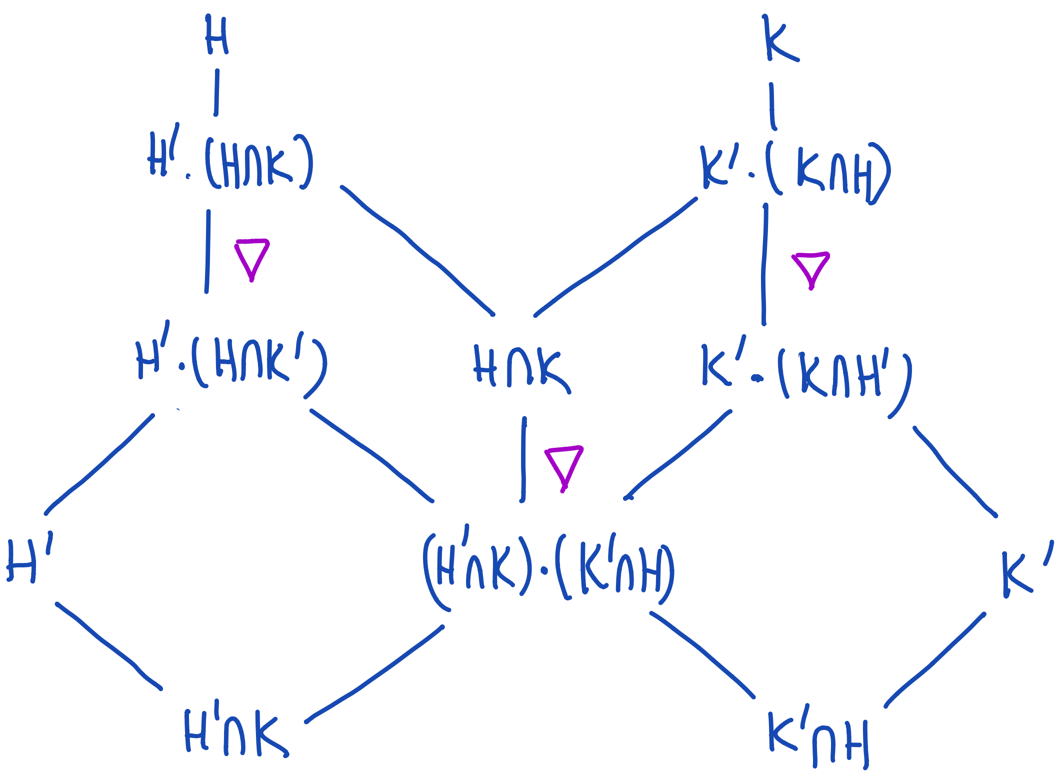
- Normality : Use $H'\cap K \lhd H\cap K$
  $K'\cap H \lhd H\cap K$.

$$g(H'\cap K)\cdot(K'\cap H)g^{-1} = g(H'\cap K)g^{-1}g(K'\cap H)g^{-1}$$
$$\subseteq (H'\cap K)(K'\cap H) \text{ for each } g\in H\cap K$$

so $(H'\cap K)\cdot(K'\cap H) \lhd H\cap K$.

Guiding Picture (statement)

H, K, H', K' $\quad H' \lhd H \quad K' \lhd K$



$$\frac{H'\cdot(H\cap K)}{H'\cdot(H\cap K')} \simeq \frac{K'\cdot(H\cap K)}{K'\cdot(H'\cap K)} \simeq \frac{H\cap K}{(H'\cap K)(K'\cap H)}$$

STEP 2 $\boxed{H'(H\cap K') \lhd H'(H\cap K)}$

Take $\mathcal{G}=H$, $N=H'$, $G_1 := H\cap K$, $G_2 = H\cap K'$ in the next Lemma.

Lemma: If $\mathcal{G}$ is a group, $G_1 < \mathcal{G}$, $N \lhd \mathcal{G}$ & $G_2 \lhd G_1$, then:

$$N\cdot G_2 \lhd N\cdot G_1$$

(& both are groups)

Pf/ By 3rd Iso: $\begin{cases} G_2\cdot N = N\cdot G_2 < \mathcal{G} \\ G_1 N = N G_1 < \mathcal{G} \end{cases}$

$$g_1 n (NG_2)(g_1 n)^{-1} = g_1 n N G_2 n^{-1} g_1^{-1}$$

$$= \underbrace{g_1 n g_1^{-1}}_{\in N}\, \underbrace{g_1 N g_1^{-1}}_{= N}\, \underbrace{g_1 G_2 g_1^{-1}}_{= G_2}\, \underbrace{(g_1 n^{-1} g_1^{-1})}_{\in N}$$

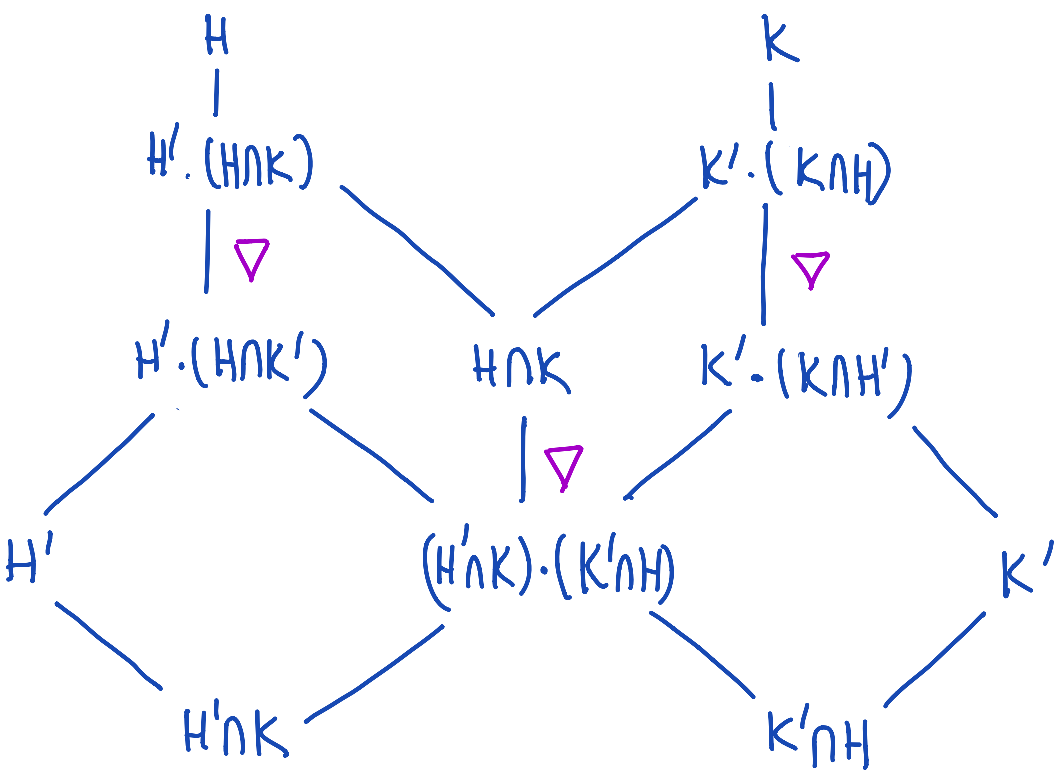
$$\in N\cdot G_2 \cdot N = N\cdot N\cdot G_2 = N\cdot G_2 .$$

for all $g_1 \in G_1$, $n\in N$. so $NG_2 \lhd N\cdot G$

$$\left[\frac{\tilde{H}N}{N} \simeq \frac{\tilde{H}}{\tilde{H}\cap N} \quad \begin{matrix} N \lhd G \\ \tilde{H} < G \end{matrix}\right]$$

Guiding Picture (statement)

H, K, H', K' $\quad H' \lhd H \quad K' \lhd K$



$$\frac{H'\cdot(H\cap K)}{H'\cdot(H\cap K')} \simeq \frac{K'\cdot(H\cap K)}{K'\cdot(H'\cap K)} \simeq \frac{H\cap K}{(H'\cap K)(K'\cap H)}$$

STEP 3 Use 3rd Iso. Thm

$$\frac{H'(H\cap K)}{H'(H\cap K')} \simeq \frac{H\cap K}{(H\cap K)\cap(H'\cdot(H\cap K'))}$$

with $N = H'\cdot(H\cap K') \lhd H$

$\tilde{H} = H\cap K$

- Symmetry on H- & K-sides gives

$$\frac{K'(H\cap K)}{K'(H'\cap K)} \simeq \frac{H\cap K}{(H\cap K)\cap(K'\cdot(H'\cap K))}$$

- To finish, we show:

$$\boxed{(H\cap K)\cap(H'\cdot(H\cap K')) = (H'\cap K)\cdot(H\cap K')}$$

- Symmetry gives:

$$(H\cap K)\cap(K'\cdot(H'\cap K)) = (H'\cap K)\cdot(H\cap K')$$

Claim: $(H\cap K)\cap(H'\cdot(H\cap K')) = (H'\cap K)\cdot(K'\cap H)$ if $H'\triangleleft H$, $K'\triangleleft K$

Pf/ $(H'\cap K)(K'\cap H) \subset (H\cap K)\cap(H'\cdot(H\cap K'))$ is clear

Conversely, let $x = a\cdot b \in H'\cdot(H\cap K')\cap(H\cap K)$ with $a\in H'$, $b\in H\cap K' \subseteq H\cap K$

$\Rightarrow a = xb^{-1} \in (H\cap K)\cdot(H\cap K) \leq H\cap K$

$\Rightarrow a\in H'\cap(H\cap K) = H'\cap K$.

Thus, $x = ab \in (H'\cap K)(H\cap K')$. $\square$

Next Time: Study composition series where graded pieces are simple groups.
(Jordan-Hölder series).