Lecture 12: Jordan-Hölder & Derived Series

Last time: Discussed composition series

- A composition series of a group G is a finite sequence of subgroups of G

$$\Sigma: \qquad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = \{e\}$$

such that $G_{j+1} \triangleleft G_j$ is normal for all $j = 0, \dots, k-1$.

- Graded pieces: $gr_i(G) := G_i / G_{i+1}$ $\quad 0 \le i \le k-1$.

- Refinement: add terms to the composition series while remaining one

- Equivalence:
  - Same number of terms
  - — graded pieces, counted with multiplicity (up to permutation)

Theorem (Schreier) Any two composition series of a group G have a "common refinement", up to equivalence.

Lemma (Zassenhaus). Fix a group G, H, K two subgroups of G & $H' \triangleleft H$, $K' \triangleleft K$. Then:

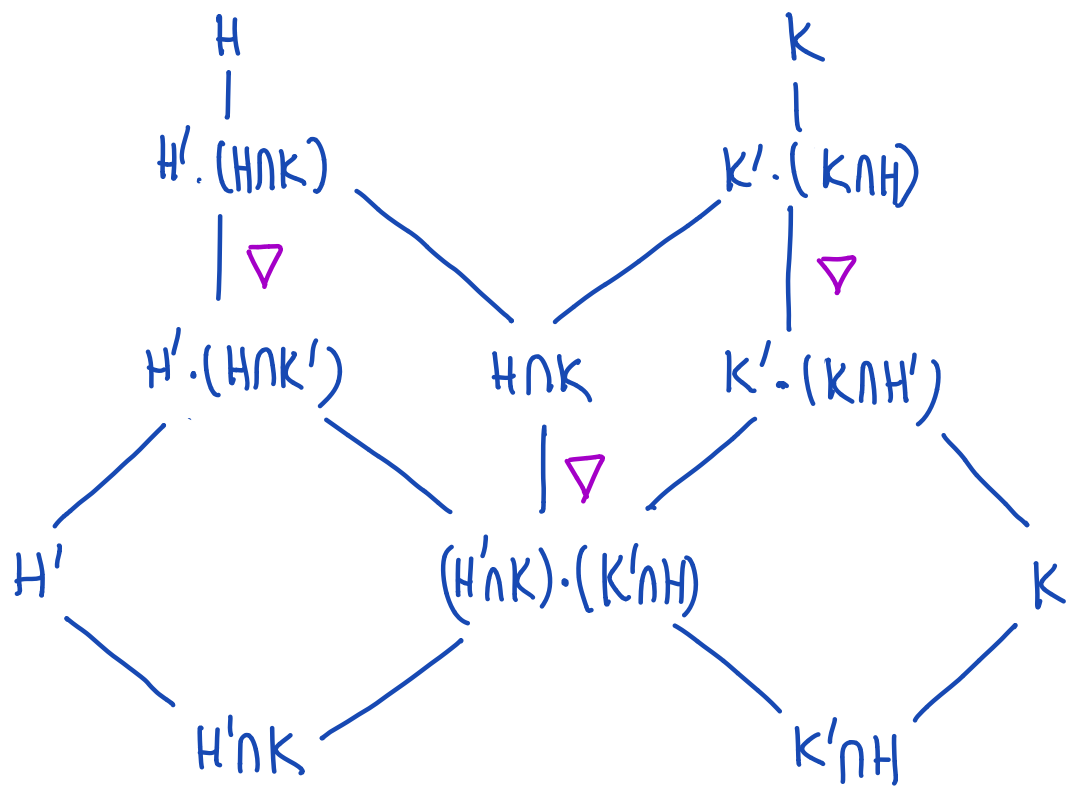
(i) $H' \cdot (H \cap K') \triangleleft H' \cdot (H \cap K)$

$K' \cdot (H' \cap K) \triangleleft K' \cdot (H \cap K)$

(Also $(H' \cap K)(K' \cap H) \triangleleft H \cap K$)

(ii) $\dfrac{H' \cdot (H \cap K)}{H' \cdot (H \cap K')} \simeq \dfrac{K' \cdot (H \cap K)}{K' \cdot (H' \cap K)}$

(Both iso to $\dfrac{H \cap K}{(H' \cap K)(K' \cap H)}$)



TODAY: Discuss maximally refined comp. series = Jordan Hölder series.

- Special series build out of commutators = Derived series.

Jordan-Hölder Series

<u>Definition</u> A composition series Σ: $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$ is said to be a <u>Jordan-Hölder series</u> if:

(i) Σ is strictly decreasing (ie $G_{j+1} \subsetneq G_j \quad \forall j = 0, \ldots, n-1$)

(ii) There is no strictly decreasing composition series distinct from Σ and finer than Σ.

<u>Proposition:</u> A composition series Σ of G is Jordan-Hölder (or JH for short) if and only if $gr_i^{\Sigma}(G)$ is simple for all $i = 0, \ldots, n-1$.

(<u>Recall</u>: $\{e\}$ is not simple; G is simple if $H \trianglelefteq G \Rightarrow H = \{e\}$ or G)

$$\boxed{\text{JH series} \iff \text{simple graded pieces}}$$

⚠ A general group G need NOT posses a JH series

Proposition. Every finite group G has a Jordan-Hölder series.

Theorem (Jordan-Hölder) Two Jordan-Hölder series of a group G are equivalent.

Corollary: Let G be a group that admits a JH series. If Σ is any strictly decreasing composition series of G, then there exists a JH series refining Σ.

EXAMPLES

Derived Series of a group (Commutator Series)

Recall: $[G:G] = \langle \underbrace{aba^{-1}b^{-1}}_{=:[a:b]} : a, b \in G \rangle$ commutator subgroup of G.

Definition: given $A, B < G$, we consider
$(A:B) = \langle aba^{-1}b^{-1} : a \in A, b \in B \rangle$

Lemma: If $A, B \lhd G$, then $(A, B) \lhd G$.

We will use commutators to define a potential composition series for G

Recursive: $D^0(G) = G$, $D^{n+1}(G) = D(D^n(G)) := (D^n(G), D^n(G))$

- $D^{n+1}(G) \lhd D^n(G)$ & $D^n(G)/D^{n+1}(G)$ is abelian (Problem 11 HW1)

Def: $\mathcal{D}$: $G = D^0(G) \geq D^1(G) \geq \cdots\cdots$ derived series for G

$$\boxed{D^0(G) = G, \quad D^{n+1}(G) = D(D^n(G)) := (D^n(G), D^n(G)) \lhd D^n(G)}$$

Q: When is this a composition series?

Definition: We say G is solvable if $\exists\, N \geq 0$ with $D^N(G) = \{e\}$

Remarks: ① The term "solvable" originates from Galois Theory (Math 6112)

② $D^0(G) = \{e\} \iff G$ is

③ $D^1(G) = \{e\} \iff G$ is

Proposition: If G is non-abelian & simple, then $D^n(G) = G$ for all $n \geq 0$, so G is not solvable.

$\boxed{\text{EXAMPLES}}$

Q: JH series refining $D_n = \langle s, \rho \rangle \supseteq \langle \rho^2 \rangle \supseteq \{e\}$?