Lecture 12: Jordan-Hölder & Derived Series

Last time: Discussed composition series

- A composition series of a group G is a finite sequence of subgroups of G

$$\Sigma: \qquad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = \{e\}$$

such that $G_{j+1} \lhd G_j$ is normal for all $j = 0, \ldots, k-1$.

- Graded pieces: $gr_i(G) := G_i / G_{i+1}$ $\quad 0 \le i \le k-1$.

- Refinement: add terms to the composition series while remaining one

- Equivalence:
  - Same number of terms
  - — graded pieces, counted with multiplicity (up to permutation)

Theorem (Schreier) Any two composition series of a group G have a "common refinement", up to equivalence.

Lemma (Zassenhaus). Fix a group G, H, K two subgroups of G & $H' \triangleleft H$, $K' \triangleleft K$. Then:

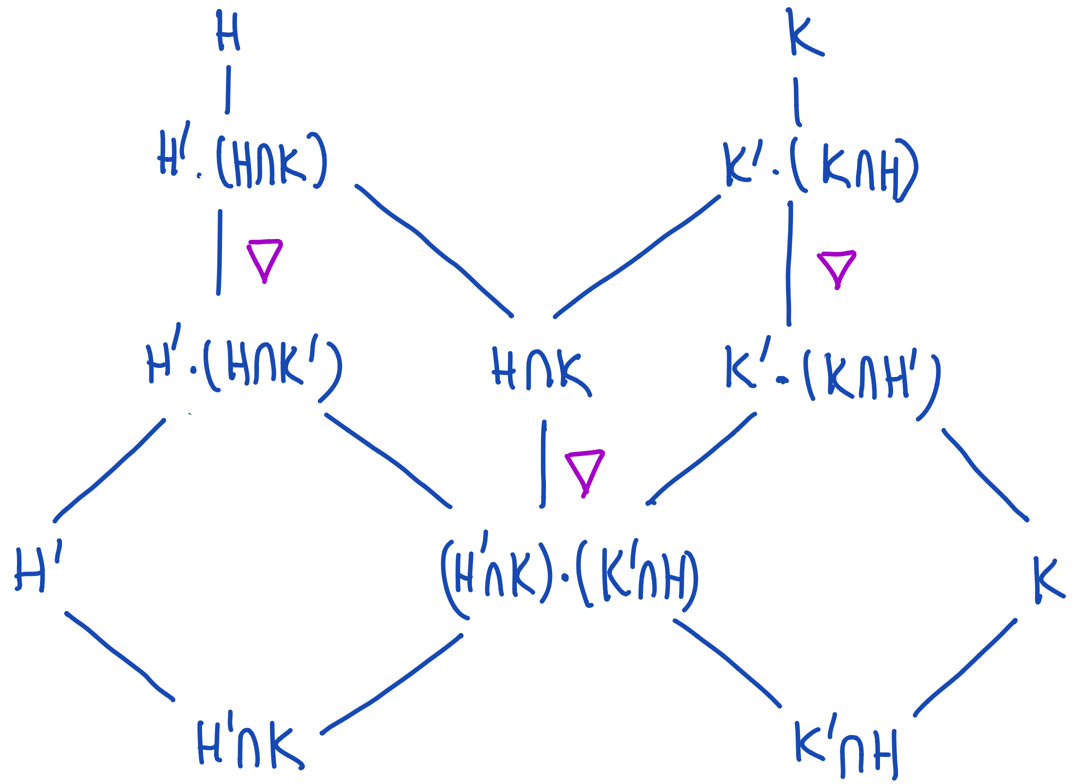
(i) $H' \cdot (H \cap K') \triangleleft H' \cdot (H \cap K)$

$K' \cdot (H' \cap K) \triangleleft K' \cdot (H \cap K)$

(Also $(H' \cap K)(K' \cap H) \triangleleft H \cap K$)

(ii) $$\frac{H' \cdot (H \cap K)}{H' \cdot (H \cap K')} \simeq \frac{K' \cdot (H \cap K)}{K' \cdot (H' \cap K)}$$

(Both iso to $\dfrac{H \cap K}{(H' \cap K)(K' \cap H)}$)



TODAY: . Discuss maximally refined comp. series = Jordan Hölder series.

. Special series build out of commutators = Derived series.

Jordan-Hölder Series

<u>Definition</u> A composition series $\Sigma: G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$ is said to be a <u>Jordan-Hölder series</u> if:

(i) Σ is strictly decreasing (ie $G_{j+1} \subsetneq G_j$ $\forall j = 0, \ldots, n-1$)

(ii) There is no strictly decreasing composition series distinct from Σ and finer than Σ.

<u>Example</u>: $\left.\begin{array}{l} \Sigma_1 : G = \mathbb{Z}/6\mathbb{Z} \supsetneq \mathbb{Z}/3\mathbb{Z} \supsetneq \{e\} \\ \Sigma_2 : G = \mathbb{Z}/6\mathbb{Z} \supsetneq \mathbb{Z}/2\mathbb{Z} \supsetneq \{e\} \end{array}\right\}$ are JH series

$$gr_0^{\Sigma_1}(G) = \mathbb{Z}/2\mathbb{Z} = gr_1^{\Sigma_2}(G) \quad \& \quad gr_1^{\Sigma_1}(G) = \mathbb{Z}/3\mathbb{Z} = gr_0^{\Sigma_2}(G).$$

<u>Proposition</u>: A composition series Σ of G is Jordan-Hölder (or JH for short) if and only if $gr_i^{\Sigma}(G)$ is simple for all $i = 0, \ldots, n-1$.

(<u>Recall</u>: $\{e\}$ is not simple; G is simple if $H \trianglelefteq G \Rightarrow H = \{e\}$ or G)

JH series $\iff$ simple graded pieces

Pf/. Σ comp series is strictly decreasing $\iff$ no trivial ($=\{e\}$) graded pieces

($\Leftarrow$) Let $\Sigma: G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_n = \{e\}$ be a strictly decreasing comp series, but <u>NOT</u> JH, so we can refine it to Σ'

$\Rightarrow \exists\ i = 0, \dots, n-1$ with $G_{i+1} \ntrianglelefteq G_i$ are not consecutive in Σ'. Thus,

$$G_{i+1} \ntrianglelefteq H_k \trianglelefteq \cdots H_2 \ntrianglelefteq H_1 \ntrianglelefteq G_i \quad \text{part of } \Sigma'$$

In particular, $G_{i+1} \trianglelefteq H_1$ since $G_{i+1} \trianglelefteq G_i$ & $G_{i+1} < H_1 < G_i$.

Hence, H_1/G_{i+1} is a nontrivial normal subgroup of G_i/G_{i+1}, so $gr_i(G)$ is not simple. <u>Contr!</u>

($\Rightarrow$) Pick Σ comp series with some $gr_i^\Sigma(G)$ not simple.

- If $gr_i^\Sigma(G) = \{e\}$ then Σ is not strictly decreasing, so not JH.
- If $gr_i^\Sigma(G) \neq \{e\}$ & not simple, $\exists\ G_{i+1} < H < G_i$ with $H \trianglelefteq G_i$

$\left(\{e_{G_i/G_{i+1}}\} \subseteq \overline{H} \trianglelefteq G_i/G_{i+1} \qquad H := \pi^{-1}(\overline{H}) \quad \pi: G_i \twoheadrightarrow G_i/G_{i+1}\right)$

So $G_{i+1} \trianglelefteq H \trianglelefteq G$ &

$$\Sigma': G = G_0 \geq G_1 \geq \cdots \geq G_i \trianglerighteq H \trianglerighteq G_{i+1} \trianglerighteq \cdots \geq \{e\}$$

is finer than Σ, so Σ is not JH.

⚠ A general group G need NOT posses a JH series

Ex. $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq 8\mathbb{Z} \supsetneq \cdots \supsetneq G_k = 2^k\mathbb{Z} \supsetneq \cdots$ cannot terminate

Proposition. Every finite group G has a Jordan-Holder series.

Pf/ By induction on $|G|$

- If G is simple : $G \supseteq \{e\}$ is JH.
- If G is not simple : $\exists\ H \underset{\neq}{\trianglelefteq} G$ maximal. so G/H is simple

  $|H| < G$ so $\exists$ $H = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_k = \{e\}$ JH series

  So $G = G_0 \supseteq G_1 = H \supseteq G_2 = H_1 \supsetneq \cdots \supseteq G_{k+1} = \{e\}$ is JH.

□

Ex. $G = \mathbb{Z}/6\mathbb{Z}$

$\Sigma_1 : \mathbb{Z}/6\mathbb{Z} \supsetneq \mathbb{Z}/3\mathbb{Z} \supsetneq \{e\}$ JH

$\Sigma_2 : \mathbb{Z}/6\mathbb{Z} \supsetneq \mathbb{Z}/2\mathbb{Z} \supsetneq \{e\}$ JH

Same graded pieces, ie equivalent.

Q. Is this always the case? A: YES!

Theorem (Jordan-Hölder) Two Jordan-Hölder series of a group G are equivalent.

Proof: Let Σ_1, Σ_2 be two JH series of G. By Schrier's Thm, we can refine them to Σ_1' & Σ_2' where Σ_1' & Σ_2' are equivalent.

As Σ_1 (and Σ_2) is JH, Σ_1' (and Σ_2') is either identical to Σ_1 (resp. Σ_2) or it is obtained from Σ_1 (resp. Σ_2) by repeating some terms. As the series of quotients of Σ_1', & Σ_2' differ only in the order of the graded pieces, after removing all trivial quotients, the same is true for Σ_1 & Σ_2 $\square$

Corollary: Let G be a group that admits a JH series. If Σ is any strictly decreasing composition series of G, then there exists a JH series refining Σ.

Sketch of a proof: Let Σ_0 be a J-H series of G. By Schrier's Thm, $\exists\ \Sigma_0', \Sigma$ two equivalent composition series refining Σ_0 & Σ, resp. Proof of JH Thm $\Rightarrow \Sigma_0'$ JH So Σ is also JH.

EXAMPLES

① $G = \mathbb{Z}/p^k\mathbb{Z}$ with $k > 1$, write $G = \langle g \rangle$

If Σ JH $\Rightarrow$ graded pieces = p-groups, nontrivial, simple $\Rightarrow \simeq \mathbb{Z}/p\mathbb{Z}$

$$\Rightarrow \Sigma : G = G_0 \supseteq \underset{\langle g^p \rangle}{\overset{\|}{G_1}} = \mathbb{Z}/p^{k-1}\mathbb{Z} \supseteq \underset{\langle g^{p^2} \rangle}{\overset{\|}{G_2}} = \mathbb{Z}/p^{k-2}\mathbb{Z} \supseteq \cdots \supseteq \underset{\langle g^{p^{k-1}} \rangle}{\overset{\|}{G_{k-1}}} = \mathbb{Z}/p\mathbb{Z} \supseteq G_k = \{e\}$$

② $G = \mathbb{Z}/n\mathbb{Z}$ How to build JH series?

<u>n prime</u> $G \supseteq \{e\}$ is JH.

<u>n not prime</u> Write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$

$G = \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ $\quad m = p_2^{a_2} \cdots p_k^{a_k}$

<u>Induction</u>: Have a JH series for $\mathbb{Z}/m\mathbb{Z}$

$$H = \mathbb{Z}/m\mathbb{Z} \supset H_1 \supset \cdots \supset H_\ell = \{e\}$$

Example 1 gives a JH series for $\mathbb{Z}/p_1^{a_1}\mathbb{Z}$

$$\mathbb{Z}/p_1^{a_1}\mathbb{Z} = K_0 \supsetneq K_1 \supsetneq K_2 \supsetneq \cdots \supset K_s = \{e\}$$

$$\Rightarrow G_j = \begin{cases} K_j \times \mathbb{Z}/m\mathbb{Z} & \text{for } j = 0, \dots, s \\ \{e\} \times H_{j-s} & \text{for } j = s+1, \dots, \ell+s \end{cases}$$

gives JH series for $\mathbb{Z}/n\mathbb{Z}$.

Derived Series of a group (Commutator Series)

Recall: $[G:G] = \langle \underbrace{aba^{-1}b^{-1}}_{=:[a:b]} : a,b \in G \rangle$ commutator subgroup of G.

Definition: given $A, B < G$, we consider
$$(A:B) = \langle aba^{-1}b^{-1} : a \in A,\ b \in B \rangle$$

Lemma: If $A, B \triangleleft G$, then $(A,B) \triangleleft G$.

Proof: For all $g \in G$, $a \in A$, $b \in B$:
$$g\,aba^{-1}b^{-1}g^{-1} = \underbrace{(gag^{-1})}_{\in A}\underbrace{(gbg^{-1})}_{\in B}(gag^{-1})^{-1}(gbg^{-1})^{-1} = [\overset{\in A}{gag^{-1}} : \overset{\in B}{gbg^{-1}}] \in (A,B) \qquad \square$$

We will use commutators to define a potential composition series for G

Recursive: $D^0(G) = G$, $\quad D^{n+1}(G) = D(D^n(G)) := (D^n(G), D^n(G))$

- $D^{n+1}(G) \triangleleft D^n(G)$ & $D^n(G)/D^{n+1}(G)$ is abelian (Problem 11 HW1)

Def: $\mathcal{D}$: $\quad G = D^0(G) \geq D^1(G) \geq \cdots\cdots$ <u>derived series</u> for G

$$\boxed{D^0(G) = G, \quad D^{n+1}(G) = D(D^n(G)) := (D^n(G), D^n(G)) \lhd D^n(G)}$$

Q: When is this a composition series? **A:** Need $D^N(G) = \{e\}$ for some N.

Definition: We say G is solvable if $\exists\, N \geq 0$ with $D^N(G) = \{e\}$

Remarks: ① The term "solvable" originates from Galois Theory (Math 6112)

② $D^0(G) = \{e\} \iff G$ is trivial.

③ $D^1(G) = \{e\} \iff G$ is abelian (hence, all abelian groups are solvable)

Proposition: If G is non-abelian & simple, then $D^n(G) = G$ for all $n \geq 0$, so G is not solvable.

Pf/ $D(G) \neq \{e\}$ (otherwise, G would be abelian), $D(G) \lhd G$ normal $\Rightarrow D(G) = G$. $\Bigg\} \Rightarrow D^n(G) = D^0(G) = G$ for all n.

EXAMPLES

① $G = D_n$, $D^0_n = \langle \rho^2 \rangle$ which is abelian.

Jt/. $[s,\rho] = s\rho s^{-1}\rho^{-1} = s\rho s\rho^{-1} = \rho^{-2}$ · $[\rho^i : \rho^j] = e$

· $[s\rho^i : \rho^j] = s\rho^i\rho^j(s\rho^i)^{-1}\rho^{-j} = s\rho^{i+j}\rho^{-i}s\rho^{-j} = \rho^{-2j}$

· $[s\rho^i : s\rho^j] = s\rho^i s\rho^j (s\rho^i)^{-1}(s\rho^j)^{-1} = \rho^{j-i}\rho^{-i}s\rho^{-j}s$

$$= \rho^{j-2i}\rho^j = \rho^{2(j-i)}.$$

So $\mathcal{D}: G = D_n \supseteq D^0(G) = \langle \rho^2 \rangle \supseteq D^2(G) = \{e\}$ & $\boxed{D_n \text{ is solvable}}$

· $\exists$ JH series refining the derived series $\mathcal{D}$:

$$gr_0^{\mathcal{D}}(D_n) = \frac{\langle \rho, s \rangle}{\langle \rho^2 \rangle} \quad \& \quad gr_1^{\mathcal{D}}(D_n) = \frac{\langle \rho^2 \rangle}{\langle e \rangle}$$

$\Longrightarrow$ The answer depends on the parity of n!

Q: JH series refining $D_n = \langle s, \rho \rangle \supseteq \langle \rho^2 \rangle \supseteq \{e\}$?

If n is odd: $\langle \rho^2 \rangle = \langle \rho \rangle \simeq \mathbb{Z}/n\mathbb{Z}$

$\operatorname{gr}_0^{\mathcal{D}}(D_n) = \dfrac{\langle \rho, s \rangle}{\langle \rho \rangle} \simeq \mathbb{Z}/2\mathbb{Z}$ (order is $\frac{2n}{n} = 2$) simple

$\operatorname{gr}_1^{\mathcal{D}}(D_n) = \langle \rho \rangle \simeq \mathbb{Z}/n\mathbb{Z}$ not simple if n is not prime

$\rightsquigarrow$ We refine $\langle \rho \rangle = \mathbb{Z}/n\mathbb{Z} \supseteq \langle e \rangle$ to a JH-series using Example 2.

If n is even: $\langle \rho^2 \rangle \simeq \mathbb{Z}/\frac{n}{2}\mathbb{Z}$ $n = 2m$

- $|\operatorname{gr}_0^{\mathcal{D}}(D_n)| = \dfrac{2n}{m} = 4 = 2^2 \Longrightarrow \operatorname{gr}_0^{\mathcal{D}}(D_n) \simeq \mathbb{Z}/4\mathbb{Z}$ or $\boxed{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$

  Classification of p^2-order gps $\quad \langle \bar{\rho} \rangle \times \langle \bar{s} \rangle$

- $\operatorname{gr}_1^{\mathcal{D}}(D_n) \simeq \mathbb{Z}/m\mathbb{Z}$ $\rightsquigarrow$ can be refined to JH series
- We refine $\langle s, \rho \rangle \supseteq \langle \rho^2 \rangle$ by $\langle s, \rho \rangle \supseteq \langle \rho \rangle \supseteq \langle \rho^2 \rangle$

  $\dfrac{\langle s, \rho \rangle}{\langle \rho \rangle} \simeq \langle s \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ & $\dfrac{\langle \rho \rangle}{\langle \rho^2 \rangle} \simeq \mathbb{Z}/2\mathbb{Z}$ simple

- Combine the 2 refinement to get JH refining $\mathcal{D}(D_{2m})$.