Lecture 13: Solvable & Nilpotent Groups

Recall: We defined the derived series of a group G via

$(A:B) = \langle [a,b] = aba^{-1}b^{-1} : a \in A, b \in B \rangle \lhd G$ if $A, B \lhd G$

- $D^0(G) = G$; $D^{n+1}(G) = (D^n(G), D^n(G))$
- Main property: $H/(H,H)$ is abelian, hence any subgroup of H containing (H,H) is normal. Conversely, if $A \lhd H$ & H/A is abelian, then $(H,H) \subseteq A$.
- Solvable groups: $D^N(G) = \{e\}$ for some $N \geq 0$.

Equiv: $\mathcal{D}: G \supseteq D(G) \supseteq D^2(G) \supseteq \dots \supseteq D^N(G) = \{e\}$ is a composition series for G & $D^j(G)/D^{j+1}(G)$ is abelian $\forall j$

Examples: ① $G = \{e\}$ ($D^0(G) = \{e\}$)

② G abelian ($D^1(G) = (G,G) = \{e\}$)

③ D_n $D^1(D_n) = \langle \rho^2 \rangle$ abelian, so $D^2(D_n) = \{e\}$

More examples

Lemma: The group of upper triangular invertible matrices is solvable.

Eg: $B := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad a, d \in \mathbb{C}^\times, b \in \mathbb{C} \right\}$

$\cup$

$D(B) =$

$\cup$

$D^2(B) =$

<u>Recall</u> If G is non-abelian & simple, then $D^n(G) = G$ for all $n \geq 0$, so G is not solvable.

<u>Ex.</u>: $D^0(S_n) = A_n$ and A_n is simple nonabelian for $n \geq 0$, so S_n is not solvable for $n \geq 5$.

Pf/

<u>Obs</u>: This will be used to show that quintic or higher degree polynomials cannot be solved by radicals (like the quadratic polynomials in $\mathbb{C}[x]$)

Theorem 1: Let G be a group, and assume it has a composition series

$$\Sigma: \; G = G_0 \geq G_1 \geq \dots \geq G_n = \{e\}$$

where each $gr_i(G) = G_i / G_{i+1}$ is abelian. Then, G is solvable.

Corollary: G is solvable $\iff$ $\exists$ comp series with abelian graded pieces.

Theorem 2: Let G be a p-group. Then, there exists a comp series for G

$$G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_r = \{e\} \text{ with } gr_j(G) = G_j / G_{j+1} \simeq \mathbb{Z}/p\mathbb{Z} \quad \forall i$$

Corollary 2: Every p-group is solvable.

Proposition: Let G be a group & $N \lhd G$. Then, G is solvable if, and only if, N & G/N are.

Equivalently: $\mathbb{1} \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \mathbb{1}$ ses

Then G_2 is solvable $\Longleftrightarrow$ G_1 & G_3 are solvable.

$$\boxed{G \text{ solvable} \Longleftrightarrow N \triangleleft G \text{ \& } G/N \text{ are}}$$

Proof: ($\Longrightarrow$)

$$\boxed{G \text{ solvable} \Leftarrow N \triangleleft G \ \& \ G/N \text{ are}}$$

$(\Leftarrow)$

Q: What can we say about Jordan–Hölder series of finite, solvable gps?

Proposition: Fix G a finite group. Then, the following are equivalent:

(1) G is solvable

(2) $gr_j^{\Sigma}(G)$ is cyclic of prime order $\forall j$ for some JH series Σ of G.

(3) $gr_j^{\Sigma}(G)$ is cyclic of prime order $\forall j$ for ALL JH series Σ of G.

Proof:

Lower Central Series

We now define a new sequence involving commutators

Set $\boxed{C^1(G) = G \ \& \ C^{n+1}(G) = (G, C^n(G)) \quad \forall n \geq 1}$

By induction on n we see $C^n(G) \lhd G$ $\forall n$ $\left(C^{n+1}(G) \lhd G \text{ if } C^n(G) \lhd G\right)$

<u>Lemma</u>: $C^{n+1}(G) < C^n(G)$ so $C^{n+1}(G) \lhd C^n(G)$

Pf/

$\rightsquigarrow$ New sequence : $\boxed{\mathscr{C}: \quad G = C^1(G) \geq C^2(G) \geq C^3(G) \geq \ldots\ldots}$

<u>Definition</u>: G is <u>nilpotent</u> if $\exists\, n \geq 1$ such that $C^n(G) = \{e\}$.

Equivalently, $\mathscr{C}$ is a composition series for G.