Lecture 13: Solvable & Nilpotent Groups

Recall: We defined the derived series of a group G via

$(A:B) = \langle [a,b] = aba^{-1}b^{-1} : a \in A, b \in B \rangle \trianglelefteq G$ if $A, B \trianglelefteq G$

- $D^0(G) = G$; $D^{n+1}(G) = (D^n(G), D^n(G))$
- Main property: $H/(H,H)$ is abelian, hence any subgroup of H containing (H,H) is normal. Conversely, if $A \trianglelefteq H$ & H/A is abelian, then $(H,H) \subseteq A$.
- Solvable groups: $D^N(G) = \{e\}$ for some $N \geq 0$.

Equiv: $\mathcal{D}: G \geq D(G) \geq D^2(G) \geq \ldots\ldots \geq D^N(G) = \{e\}$ is a composition series for G & $D^j(G)/D^{j+1}(G)$ is abelian $\forall j$

Examples: ① $G = \{e\}$ ($D^0(G) = \{e\}$)

② G abelian ($D^1(G) = (G,G) = \{e\}$)

③ D_n $D^1(D_n) = \langle \rho^2 \rangle$ abelian, so $D^2(D_n) = \{e\}$

More examples

Lemma: The group of upper triangular invertible matrices is solvable.

Eg: $B := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad a, d \in \mathbb{C}^\times, \; b \in \mathbb{C} \right\}$

$$\cup$$

$$D(B) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{C} \right\}$$

$$\cup$$

$$D^2(B) = \{e\}$$

Pf/ . Assume $D(B)$ is as claimed, then $D(B) \simeq \mathbb{C}$ abelian
$\Rightarrow D^2(B) = \{e\}$.

. We argue the claim for $D(B)$ holds by explicit computation

$$\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \right] = \frac{1}{ad\,a'd'} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} d' & -b' \\ 0 & a' \end{pmatrix}$$

$$= \frac{1}{ada'd'} \begin{pmatrix} aa' & b'a + bd' \\ 0 & dd' \end{pmatrix} \begin{pmatrix} dd' & -b'd - ba' \\ 0 & aa' \end{pmatrix}$$

$$= \frac{1}{ada'd'} \begin{pmatrix} aa'dd' & aa'(b'a + bd' - b'd - ba') \\ 0 & dd'aa' \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ for } x = \frac{1}{dd'}\left(b'(a-d) - b(a'-d')\right)$$

<u>Check</u>: any $x \in \mathbb{C}$ is achieved $\Rightarrow$ Claim follows.

Recall If G is non-abelian & simple, then $D^n(G) = G$ for all $n \geq 0$, so G is not solvable.

Ex: $D^0(S_n) = A_n$ and A_n is simple nonabelian for $n \geq 0$, so S_n is not solvable for $n \geq 5$.

Pf/ Assume A_n is simple for $n \geq 5$. Then

- $[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1} \in A_n$ for all σ, τ $\Rightarrow$ $D^0(S_n) \subseteq A_n$
- Since S_n is not abelian $D^0(S_n) \neq \{e\}$
- $D^0(S_n) \triangleleft S_n$ $\Longrightarrow$ $\underset{\{e\}}{\underset{\nparallel}{D^0(S_n)}} \triangleleft A_n$ forces $D^0(S_n) = A_n$.

Obs: This will be used to show that quintic or higher degree polynomials cannot be solved by radicals (like the quadratic polynomials in $\mathbb{C}[x]$)

Theorem 1: Let G be a group, and assume it has a composition series

$$\Sigma: \ G = G_0 \geq G_1 \geq \dots \geq G_n = \{e\}$$

where each $gr_i(G) = G_i / G_{i+1}$ is abelian. Then, G is solvable.

Proof: We will show $D^j(G) \subseteq G_j \ \forall j$ by induction on j. In particular $D^n(G) \subseteq \{e\}$ so $D^n(G) = \{e\}$ & G is solvable.

Base case: $j=0$ is clear since $D^0(G) = G = G_0$.

Inductive Step: Assume $D^j(G) \subset G_j$. Since G_j/G_{j+1} is abelian, then $D(G_j) \subseteq G_{j+1}$. But $D(G_j) = D^{j+1}(G)$

So $D^{j+1}(G) = D(D(G_j)) \subseteq D(G_j) \subseteq G_{j+1}$, as we wanted $\square$

Corollary: G is solvable $\iff \exists$ comp series with abelian graded pieces.

Theorem 2: Let G be a p-group. Then, there exists a comp series for G

$$G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_r = \{e\} \text{ with } gr_j(G) = G_j / G_{j+1} \simeq \mathbb{Z}/p\mathbb{Z} \quad \forall i$$

Proof: By induction on k where $|G| = p^k$.

- **Base case:** $k=1$ is clear since $G \simeq \mathbb{Z}/p\mathbb{Z}$ ($r=1$ will do)
- **Inductive Step:** Fix $Z = Z(G)$ center of G. (p-group) $\Rightarrow Z \neq \{e\}$

  Pick $x \in Z \setminus \{e\}$. So order$(x) = p^s$ & $s \geq 1$ & $x^{p^{s-1}}$ has order p. Set $H = \langle x^{p^{s-1}} \rangle \subset Z$ so $H \trianglelefteq G$. & $H \simeq \mathbb{Z}/p\mathbb{Z}$

  Now $|G/H| = p^{k-1}$ & by inductive hypothesis we can find a composition series for G/H: $G/H = \overline{G_0} \geq \overline{G_1} \geq \cdots \geq \overline{G_r} = \{eH\}$ with $\overline{G_i}/\overline{G_{i+1}} \simeq \mathbb{Z}/p\mathbb{Z}$ for all i. Use $\pi : G \twoheadrightarrow G/H$ to define $\boxed{G_j = \pi^{-1}(\overline{G_j})}$.

- $G_{j+1} \triangleleft G_j \;\; \forall j$ & $H < G_j \;\forall j$ & $G_j/G_{j+1} \simeq \dfrac{G_j/H}{G_{j+1}/H} = \overline{G_j}/\overline{G_{j+1}} \simeq \mathbb{Z}/p\mathbb{Z}$
- $G_r = H \simeq \mathbb{Z}/p\mathbb{Z}$; $G_0 = G$ $\rightsquigarrow$ G_i's give the series we wanted $\square$

Corollary 2: Every p-group is solvable.

Pf/ Take the composition series from Thm 2 & use that $\mathbb{Z}/p\mathbb{Z}$ is abelian $\square$

Proposition: Let G be a group & $N \lhd G$. Then, G is solvable if, and only if, N & G/N are.

Equivalently: $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$ ses
Then G_2 is solvable $\Longleftrightarrow$ G_1 & G_3 are solvable.

Example $G = D_n$, $N = \langle \rho \rangle \lhd G$ & $G/N \simeq \mathbb{Z}/2\mathbb{Z}$ both solvable (abelian),
so D_n is solvable.

$\boxed{G \text{ solvable} \iff N \trianglelefteq G \ \& \ G/N \text{ are}}$

<u>Proof</u>: ($\Rightarrow$) First, assume G is solvable & pick $n \geq 0$ with $D^n(G) = \{e\}$

Then $D^n(N) \subseteq D^n(G) = \{e\} \implies N$ is solvable.

↳ clear by easy induction $D^j(N) \subseteq D^j(G)$ for all j

If $\pi : G \twoheadrightarrow G/N$ is the natural projection, then:

$$\pi(D(G)) = \pi((G:G)) \underset{\substack{\uparrow \\ \pi \text{ group hom}}}{=} (\pi(G), \pi(G)) = D(G/N)$$

Then $\pi(D^{j+1}(G)) = D(D^j(G/N)) = D^{j+1}(G/N)$

So $D^n(G/N) = \pi(\{e\}) = \{e_{G/N}\}$ so G/N is solvable

$\boxed{G \text{ solvable} \Longleftarrow N \lhd G \ \& \ G/N \text{ are}}$

($\Leftarrow$) Now, assume N & G/N are solvable. By Theorem 1 we have composition series for N & G/N with abelian graded pieces

$$\Sigma: \quad N = N_0 \geq N_1 \geq \cdots \geq N_k = \{e\} \qquad \frac{N_i}{N_{i+1}} \text{ abelian } \forall i = 0, \ldots, k-1.$$

$$\Sigma': \quad G/N = \bar{G}_0 \geq \bar{G}_1 \geq \cdots \geq \bar{G}_s = \{e_{G/N}\} \qquad \bar{G}_j / \bar{G}_{j+1} \text{ —— } \forall j = 0, \ldots, s-1.$$

Set $\pi : G \twoheadrightarrow G/N$ & $\boxed{G_j := \pi^{-1}(\bar{G}_j)}$ $\forall j = 0, \ldots, s$

So $G_s = N$, $G_0 = G$ & $G_j / G_{j+1} \simeq \bar{G}_j / \bar{G}_{j+1}$ abelian $(N < G_j \ \forall j)$

Set $\boxed{G_{s+i} := N_i}$ for $i = 1, \ldots, k$. Then:

$$\Sigma'' \quad G = G_0 \geq G_1 \geq \cdots \geq G_s = N \geq G_{s+1} \geq \cdots \geq G_{s+k} = \{e\}$$

is a comp series for G with abelian graded pieces. By Thm 1, G is solvable. $\square$

Q: What can we say about Jordan–Hölder series of finite, solvable gps?

Proposition: Fix G a finite group. Then, the following are equivalent:

(1) G is solvable

(2) $gr_j^{\Sigma}(G)$ is cyclic of prime order $\forall j$ for some JH series Σ of G.

(3) $gr_j^{\Sigma}(G)$ is cyclic of prime order $\forall j$ for ALL JH series Σ of G.

Proof: (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (3) Assume G is solvable & pick any Jordan-Hölder series Σ_1 of G (it exists because G is finite)

Pick a comp series Σ_2 of G with abelian graded pieces ($\exists$ since G is solvable).

By Schreier's Thm $\exists$ Σ_1' & Σ_2' refinement of Σ_1 & Σ_2 with $\Sigma_1 \simeq \Sigma_2$.

- So the graded pieces of Σ_1' are either trivial or simple
- The graded pieces of Σ_2' are abelian (since we are refining Σ_1 & $gr_i^{\Sigma_1'}(G)$ ab. $\forall i$

$\Longrightarrow$ $gr_i^{\Sigma_1'}(G)$ trivial or (abelian & simple) $\forall i$ $\Rightarrow gr_i^{\Sigma_1'}(G) = \{e\}$ or $\mathbb{Z}/p_i\mathbb{Z}$

$\Sigma_1' \sim \Sigma_2'$

$\Longrightarrow$ Same is true for Σ_1 (JH so no nontrivial refinements! (p_i prime)).

Lower Central Series

We now define a new sequence involving commutators

Set $\boxed{C^1(G) = G \ \& \ C^{n+1}(G) = (G, C^n(G)) \quad \forall n \geq 1}$

By induction on n we see $C^n(G) \trianglelefteq G$ $\forall n$ $\left(C^{n+1}(G) \trianglelefteq G \text{ if } C^n(G) \trianglelefteq G\right)$

<u>Lemma</u>: $C^{n+1}(G) < C^n(G)$ so $C^{n+1}(G) \trianglelefteq C^n(G)$

Pf/ $C^{n+1}(G) = \langle \underbrace{gxg^{-1}}_{\in C^n(G)} \underset{C^n(G)}{x^{-1}} : g \in G,\ x \in C^n(G) \rangle < C^n(G)$

But $C^n(G) \trianglelefteq G$

Since $C^{n+1}(G) \trianglelefteq G$, we conclude: $C^{n+1}(G) \trianglelefteq C^n(G)$. $\square$

$\rightsquigarrow$ New sequence : $\boxed{\mathcal{C} : \quad G = C^1(G) \geq C^2(G) \geq C^3(G) \geq \cdots\cdots}$

<u>Definition</u>: G is <u>nilpotent</u> if $\exists\ n \geq 1$ such that $C^n(G) = \{e\}$.

Equivalently, $\mathcal{C}$ is a composition series for G.