Lecture 14: Nilpotent groups, Simplicity of A_n for $n \geq 5$

Recall: Lower central series

$$G = C^1(G) \triangleleft C^2(G) \triangleleft \cdots \triangleleft C^n(G) \triangleleft C^{n+1}(G) \triangleleft \cdots \quad (*)$$

where $C^1(G) = G \qquad C^{n+1}(G) = (G, C^n(G)) \ \forall n \geq 1$

Note: $C^j(G) \triangleleft G$ (by induction)

Definition: G is nilpotent if $\exists\, n \geq 1$ such that $C^n(G) = \{e\}$. (ie $(*)$ is a composition series)

EXAMPLES:

$$\boxed{C^1(G) = G \ ; \ C^{n+1}(G) = (G, C^n(G)) \ \forall n \geq 1}$$

Properties: ① $(G, C^n(G)) = C^{n+1}(G) \ \forall n \Rightarrow C^{n+1}(G) \triangleleft C^n(G)$

② $C^n(G) / C^{n+1}(G)$ is abelian $\forall n$

③ $C^2(G) = (G, G) = D^1(G)$

④ $(C^n(G), C^m(G)) \subset C^{n+m}(G)$ (Exercise HW5)

$\Rightarrow \ D^l(G) \subseteq C^{2^l}(G)$ for all $l \geq 0$.

Corollary: Nilpotent $\Rightarrow$ Solvable (Derived series is comp series)

⚠ $\not\Leftarrow$ Eg: D_3 is solvable but not nilpotent.

Theorem 1: G is nilpotent if and only if it has a composition series

$$\Sigma : \; G = G_0 \geq G_1 \geq \cdots\cdots \geq G_n = \{e\}$$

with (1) $gr_j^{\Sigma}(G) = G_j / G_{j+1}$ is abelian $\forall\, j = 0, \dots, n-1$

(2) $(G, G_j) \subset G_{j+1}$ $\forall\, j = 0, \dots, n-1$

Proposition: ① Subgroups and quotients of nilpotent groups are nilpotent
[Same proof as for solvable groups (see Lecture 13)]

② G is nilpotent if and only if there is a subgroup $A \subset Z(G)$ with G/A nilpotent.

↳ Otherwise false!

⚠ The last statement fails if A is not included in $Z(G)$,

ie $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$ ses & G_1, G_3 nilpotent $\not\Rightarrow$ G_2 is nilpotent

<u>Corollary</u>: p-groups are nilpotent.

Theorem: Only nilpotent groups are direct product of p-groups (HW5)

Key steps: ① Assume $N_1, N_2 \lhd G$ & $(G:N_1) \subset N_2 \subset N_1$ Then:

$$N_2 H \lhd N_1 H \text{ for all } H < G \qquad [HW5]$$

② **Lemma**: Let G be a nilpotent group & $H \lneq G$. Then:

$$H \subsetneq N_G(H) := \{ g \in G : \; gHg^{-1} = H \}.$$

The Alternating Group A_n

Recall $A_n =$ group of even permutations.

It was defined as the kernel of the sign morphism $S_n \longrightarrow \{\pm 1\}$

$$\sigma \longmapsto (-1)^{\ell(\sigma)}$$

(This map was unique defined by $\text{sign}((ab)) = -1 \quad \forall a \neq b$)

$$\boxed{\textbf{Theorem}: \ A_n \text{ is simple for } n \geq 5}$$

- Proof idea: need to find generators for A_n.

Lemma 1: For $n \geq 3$: A_n is generated by 3 cycles.

Pf/

Lemma 2 : For $n \geq 5$: all 3-cycles in S_n are conjugate to each other in A_n

Theorem 2: A_n is simple for $n \geq 5$.

Proof: