Lecture 14: Nilpotent groups, Simplicity of A_n for $n \geq 5$

Recall: Lower central series

$$G = C^1(G) \triangleleft C^2(G) \triangleleft \cdots \triangleleft C^n(G) \triangleleft C^{n+1}(G) \triangleleft \cdots \quad (*)$$

where $C^1(G) = G \qquad C^{n+1}(G) = (G, C^n(G)) \ \forall n \geq 1$

Note: $C^j(G) \triangleleft G$ (by induction)

Definition: G is nilpotent if $\exists\, n \geq 1$ such that $C^n(G) = \{e\}$.
(ie $(*)$ is a composition series)

EXAMPLES:

1. $G = \mathbb{Z}/6\mathbb{Z}$, $\quad C^2(G) = (G,G) = 1$ (G abelian)

2. $G = D_n \qquad C^2(G) = \langle \rho^2 \rangle$, $\quad C^3(G) = (G : \langle \rho^2 \rangle) = \langle [s\rho^i : \rho^{2j}] \rangle$

$s\rho^i \rho^{2j} (s\rho^i)^{-1} \rho^{-2j} = s\rho^{2j} s \rho^{-2j} = \rho^{-4j} \implies C^3(G) = \langle \rho^4 \rangle$

$C^4(G) = (G, \langle \rho^4 \rangle) = \langle \rho^8 \rangle \rightsquigarrow C^{m+1}(G) = \langle \rho^{2^m} \rangle \ \forall m \geq 1$

Conclude: D_n is nilpotent if and only if n is a power of 2.

$$\boxed{C^1(G) = G \quad ; \quad C^{n+1}(G) = (G, C^n(G)) \ \forall n \geq 1}$$

Properties: ① $(G, C^n(G)) = C^{n+1}(G) \ \forall n \Rightarrow C^{n+1}(G) \triangleleft C^n(G)$

② $C^n(G) / C^{n+1}(G)$ is abelian $\forall n$

(because $(C^n(G) : C^n(G)) \subseteq (G, C^n(G)) = C^{n+1}(G)$. ✓)

③ $C^2(G) = (G, G) = D^1(G)$

④ $(C^n(G), C^m(G)) \subset C^{n+m}(G)$ (Exercise HW5)

$\Longrightarrow \quad D^l(G) \subseteq C^{2^l}(G)$ for all $l \geq 0$.

Pf/ True for $l=0$ & $l=1$.

$$D^{l+1}(G) = (D^l(G), D^l(G)) \subseteq (C^{2^l}(G), C^{2^l}(G) \subseteq C^{2^l+2^l}(G) = C^{2^{l+1}}(G)$$

Corollary: Nilpotent $\Longrightarrow$ Solvable (Derived series is comp series)

⚠ $\not\Longleftarrow$ Eg: D_3 is solvable but not nilpotent.

Theorem 1: G is nilpotent if and only if it has a composition series

$$\Sigma: \; G = G_0 \geq G_1 \geq \cdots\cdots \geq G_n = \{e\}$$

with (1) $gr_j^{\Sigma}(G) = G_j / G_{j+1}$ is abelian $\forall j = 0, \ldots, n-1$

(2) $(G, G_j) \subset G_{j+1}$ $\forall j = 0, \ldots, n-1$

Remark: (2) $\Rightarrow$ (1) $\quad \left[(G_j, G_j) \subset (G, G_j) \underset{(2)}{\subseteq} G_{j+1} \Rightarrow G_j/G_{j+1} \text{ abelian} \right]$

Proof ($\Rightarrow$) Take $G_j = C^{j-1}(G)$ from lower central series.

($\Leftarrow$) It's enough to check the following

Claim: $C^{j+1}(G) \subseteq G_j$ $\forall j = 0, \ldots, n$ $\quad$ ($\Rightarrow C^{n+1}(G) \subseteq \{e\}$, so $C^{n+1}(G) = \{e\}$)

Pf/ By induction on j:

- **Base case:** $j=0$. $C^1(G) = G = G_0$.
- **Inductive step:** Fix $j>0$ & assume $C^j(G) \subseteq G_{j-1}$.

$$C^{j+1}(G) = (G, C^j(G)) \overset{[IH]}{\subseteq} (G, G_{j-1}) \overset{[(2)]}{\subseteq} G_j$$

$\square$

Proposition: ① Subgroups and quotients of nilpotent groups are nilpotent
[Same proof as for solvable groups (see Lecture 13)]

② G is nilpotent if and only if there is a subgroup $A \subset Z(G)$ with G/A nilpotent.

↳ otherwise false!

Proof: We only need to show ($\Leftarrow$). Consider $\pi : G \twoheadrightarrow G/A$
($A \triangleleft G$ because $A \subset Z(G)$) pick n with $C^n(G/A) = \{e\}$

Claim: $\pi(C^k(G)) = C^k(G/A) \quad \forall k$

Proof: By induction on k.

- **$k=1$:** $\pi(G:G) = (\pi(G):\pi(G))$
- **Inductive Step:** $C^{k+1}(G) = (G : C^k(G))$ so

$$\pi(C^{k+1}(G)) = (\pi(G) : \pi(C^k(G))) \underset{[IH]}{=} \left(\frac{G}{A} : C^k\left(\frac{G}{A}\right)\right) = C^{k+1}(G/A))$$

Then, $\pi(C^n(G)) \overset{(*)}{=} C^n(G/A) = \{e\} \Longrightarrow C^n(G) \subset A$

$\ker \pi = A$

By $A \subset Z(G)$ so $C^{n+1}(G) = (G : A) = \{e\}$. $\square$

⚠ The last statement fails if A is not included in $Z(G)$,

ie $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$ ses & G_1, G_3 nilpotent $\not\Longrightarrow$ G_2 is nilpotent

Example:

$$G_2 = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, d \in \mathbb{C}^\times \ b \in \mathbb{C} \right\}$$

$$\cup$$

$$G_2 \underset{[HW3]}{\rhd} G_1 = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{C} \right\} \simeq \mathbb{C}$$

$$G_2/G_1 = \mathbb{C}^\times \times \mathbb{C}^\times \quad \text{(diagonal entries)}$$

- G_1 & G_2/G_1 are nilpotent (see HW5)
- G_2 is solvable but not nilpotent

Corollary: p-groups are nilpotent.

Proof $Z(G) \neq \{e\}$ if $|G| = p^k$ for some k

Induction on k: $k=1 \Rightarrow G$ is abelian, so nilpotent.

Inductive Step & G not abelian, use $A = Z(G)$ nilp & $|G/Z(G)| = p^s$ $s<k$.

Theorem: Only nilpotent groups are direct product of p-groups (HW5)

Key steps: ① Assume $N_1, N_2 \trianglelefteq G$ & $(G:N_1) \subset N_2 \subset N_1$ Then:

$$N_2 H \trianglelefteq N_1 H \text{ for all } H < G \qquad \text{[HW5]}$$

② **Lemma:** Let G be a nilpotent group & $H \lneq G$. Then:

$$H \subsetneq N_G(H) := \{g \in G : gHg^{-1} = H\}.$$

Proof: Since G is nilpotent, the lower central series equals

$$G = G_0 \supseteq G_1 \supseteq \cdots\cdots \supseteq G_n = \{e\}$$

for some $n \geq 0$ & it satisfies $(G, G_j) \subset G_{j+1}$ & $G_j \trianglelefteq G$ $\forall j$.

Since $(G, G_j) \subset G_{j+1} \subset G_j$ then $G_{j+1} H \trianglelefteq G_j H$ $\forall j$

We get $G = G_0 H \supseteq G_1 H \supseteq \cdots\cdots \supseteq G_n H = H$.

Fix k to be the largest index with $G_k H \supsetneq G_{k+1} H = H$

Then $H \ntrianglelefteq G_k H$ and hence $N_G H \supset G_k H \neq H$ as we wanted.

□

The Alternating Group A_n

Recall A_n = group of even permutations.

It was defined as the kernel of the sign morphism $S_n \longrightarrow \{\pm 1\}$

$$\sigma \longmapsto (-1)^{\ell(\sigma)}$$

(This map was unique defined by $\mathrm{sign}((ab)) = -1 \quad \forall a \neq b$)

Examples $A_2 = \{e\}, \qquad A_3 \simeq \mathbb{Z}/3\mathbb{Z} = \langle (1\,2\,3) \rangle$

$$A_4 = \langle (1\,2\,3), (1\,2)(3\,4) \rangle$$

Obs: $(1\,2)(3\,4) = (1\,2)(1\,3)(1\,3)(3\,4) = (1\,3\,2)(1\,3\,4)$

$$\boxed{\text{Theorem: } A_n \text{ is simple for } n \geq 5}$$

- Proof idea: need to find generators for A_n.

Lemma 1: For $n \geq 3$: A_n is generated by 3 cycles.

Pf/ Write $H := \langle \sigma : \sigma \text{ is a 3-cycle} \rangle$

Since $(abc) = (ab)(bc)$ is even, we have that $H \leq A_n$

To finish, we show that every even permutation is a product of 3 cycles. Use

$$\left.\begin{array}{l} \text{(1) } (ac)(ac) = e \\ \text{(2) } (ac)(ab) = (abc) \\ \text{(3) } (ab)(cd) = (abc)(bcd) \end{array}\right\} \Rightarrow \text{Product of even \# of permutations is is always in } H$$

$\square$

Lemma 2: For $n \geq 5$: all 3-cycles in S_n are conjugate to each other in A_n

Pf/ Fix $\sigma = (a_1 a_2 a_3)$ & $\sigma' = (b_1 b_2 b_3)$ Pick $\gamma \in S_n$ with

$\gamma (a_1 a_2 a_3) \gamma^{-1} = (b_1 b_2 b_3)$ (Eg $\gamma \in S_n$ with $\gamma(a_i) = b_i \ \forall i$)

- If $\gamma \in A_n$ ✓. Otherwise, pick $c, d \notin \{b_1, b_2, b_3\}$, $c \neq d$. (ok, $n \geq 5$)

Then: $\underbrace{(cd)\gamma}_{\in A_n} (a_1 a_2 a_3) \underbrace{\gamma^{-1}(cd)}_{=((cd)\gamma)^{-1} \in A_n} = (b_1 b_2 b_3)$

$\square$

$\boxed{\text{Theorem 2: } A_n \text{ is simple for } n \geq 5.}$

Proof: Pick $K \lhd A_n$ with $K \neq \{id\}$. We will show that $K = A_n$ by finding a 3-cycle in it. Lemma 2 will then imply

$$A_n \overset{\text{Lemma 1}}{=} \langle \sigma : \sigma \text{ is a 3-cycle} \rangle \subset K \quad (\text{because } K \lhd A_n)$$

• Pick $\sigma \in K \setminus \{e\}$ with max $|X^\sigma|$ $\left(X^\sigma = \{x \in \{1, \ldots n\} : \sigma(x) = x\}\right)$

Claim: σ is a 3-cycle. (Write $\sigma = (a_1 a_2 a_3 \cdots) \cdots$ [prod of disjoint cycles])

CASE 1 Assume σ has a cycle of length ≥ 3 in its decomposition.

If $\sigma = (a_1\, a_2\, a_3)$ we are done. Otherwise we can find a_4, a_5 with

$$\boxed{a_4, a_5 \notin \{a_1, a_2, a_3\},\ a_4 \neq a_5,\ \sigma(a_4) \neq a_4 \ \&\ \sigma(a_5) \neq a_5}$$

Let $\tau = (a_3, a_4, a_5)$ & $\sigma' = \tau\sigma\tau^{-1}\sigma^{-1} \in K$. Then,

① $X^{\sigma'} \supset X^\sigma$

② $a_2 \in X^{\sigma'}$, $a_2 \notin X^\sigma$

(easy checks)

$$\Bigg\} \Rightarrow |X^{\sigma'}| \gneq |X^\sigma| \text{ Contra!}$$

CASE 2: All cycles in σ have length ≤ 2.

Write $\sigma = (ab)(cd) \cdots\cdots$ (we have at least 2 transpositions because $K \subset A_n$ & $\sigma \neq id$.) Pick $k \notin \{a,b,c,d\}$ (ok because $n \geq 5$). Then $\tau := (c\,d\,k)$ & $\sigma' := \tau\sigma\tau^{-1}\sigma^{-1} \in K$ satisfies:

$$\left.\begin{array}{l} ① \quad X^{\sigma'} \supset X^{\sigma} \setminus \{k\} \\ ② \quad a, b \in X^{\sigma'}, \quad a, b \notin X^{\sigma} \end{array}\right\} \Rightarrow |X^{\sigma'}| \gneq |X^{\sigma}| \text{ Contra!}$$

$\square$

Consequence: S_n is not solvable for $n \geq 5$.