Lecture 18: Local rings, nilpotent elements & rings of fractions

TODAY: Fix R to be a commutative ring.

Recall: • An ideal $\mathfrak{m} \subsetneq R$ is maximal if the only ideals of R containing $\mathfrak{m}$ are $\mathfrak{m}$ & R. (Equiv: $R/\mathfrak{m}$ is a field)

• Maximal ideals exist (Zorn's Lemma)

Def: R is a local ring if it has only one maximal ideal [Write $(R, \mathfrak{m})$]

③ $R = \mathbb{K}[\![x]\!]$ = power series in one variable over $\mathbb{K}$.

Claim: R is a ring.

Operations: • Componentwise addition

$$\left(\sum_{k\geq 0} a_k x^k\right) + \left(\sum_{k\geq 0} b_k x^k\right)$$

• Multiplication: $\left(\sum_{k\geq 0} a_k x^k\right)\left(\sum_{l\geq 0} b_l x^l\right)$

Q: Why is R local?

Obs: In our definition of $+$ & $\cdot$ on $\mathbb{K}[x]$ only finitely many operations in $\mathbb{K}$ were performed to get the coefficient of x^n once n is fixed.

Eg $\sum_{k=0}^{n} a_k b_{n-k}$ & $a_n + b_n$ gave the x^n-coeff of $\cdot$ & $+$.

- The same idea will give us a ring structure on

$$\boxed{\mathbb{K}((x))} = \text{Laurent series} = \left\{ \sum_{j=-N}^{\infty} a_j x^j \quad a_j \in \mathbb{K} \ \forall j \geq -N, \ N \in \mathbb{Z}_{\geq 0} \right\}$$

<u>Fun exercise</u>: This definition of $\cdot$ will not work for the abelian gp

$$K[\![x^{-1}, x]\!] = \left\{ \sum_{j=-\infty}^{\infty} a_j x^j \ : \ a_j \in K \ \forall j \in \mathbb{Z} \right\}$$

Characterizing Local rings

Proposition: R is local if & only if the set of all non-units of R is an ideal of R.

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Example 2: Fix $p \in \mathbb{Z}_{\geq 2}$ prime & set

$R = \{ \frac{a}{b} \in \mathbb{Q} \mid p \nmid b \text{ \& } \gcd(a,b) = 1 \}$ (usual name $\mathbb{Z}_{(p)}$)

Nilpotent elements

R = commutative ring

Def: An element $x \in R$ is called nilpotent if $\exists\ n \geq 1$ such that $x^n = 0$

Let $\mathcal{N}$ = set of all nilpotent elements of R.

Lemma: $\mathcal{N}$ is an ideal of R (called nilradical)

⚠ This fails if R is noncommutative.

Ring of Fractions

Motivation 1: Number Theory – Diophantine Equations

$$P_1(x_1,\dots,x_m),\ \dots,\ P_n(x_1,\dots,x_m) \in \mathbb{Z}[x_1,\dots,x_m]$$

Q: Find integer solutions to $P_1 = \cdots = P_n = 0$.

Motivation 2: Geometric viewpoint towards commutative rings.

Commutative ring R = [type] functions on [type] space X with values in $\mathbb{C}$ or other field.

Ideals = subsets of functions which vanish on a given subset $Y \subset X$ (closed)

Open sets $\longleftrightarrow$ non vanishing of certain functions

Def: Fix a commutative ring R & $S \subset R$. We say S is a **multiplicatively closed set** if:

(i) (ii) (iii)

- Next, we define an equivalence relation on $R \times S$:

$$\boxed{(a,s) \sim (b,t) \iff \qquad}$$

Def The ring of fractions of R relative to S, denoted by $S^{-1}R$ is the set $R \times S/\sim$ with

① Addition: $(a,s)+(b,t) =$

② Multiplication: $(a,s)\cdot(b,t) =$

③ Neutral elements: $0 =$ & $1 =$

Standard notation $(a,s) \in S^{-1}R \longleftrightarrow$ "$\frac{a}{s}$". (mimic $\mathbb{Q}$)

Exercise: • Verify that addition and multiplication formulae given above are well-defined (ie independent of class reps.)

• Check that $S^{-1}R$ is a ring with these operations