Lecture 18: Local rings, nilpotent elements & rings of fractions

TODAY: Fix R to be a commutative ring.

Recall: • An ideal $\mathfrak{m} \subsetneq R$ is maximal if the only ideals of R containing $\mathfrak{m}$ are $\mathfrak{m}$ & R. (Equiv: $R/\mathfrak{m}$ is a field)

• Maximal ideals exist (Zorn's Lemma)

Def: R is a local ring if it has only one maximal ideal [Write $(R, \mathfrak{m})$]

Examples: ① Every field is a local ring ($\mathfrak{m} = (0)$)

② $R = \mathbb{K}[x]/(x^3)$ is local when $\mathbb{K}$ is any field

Mxl ideals of $R \longleftrightarrow$ mxl ideals of $\mathbb{K}[x]$ containing (x^3),

But $\mathbb{K}[x]$ is PID, so any $\mathfrak{m} \subset \mathbb{K}[x]$ maximal equals (f) for $f \in \mathbb{K}[x]$ irreducible.

But $f \mid x^3$, so $(f) = (x)$. This is maximal in $\mathbb{K}[x]$!

$\Longrightarrow \tilde{\mathfrak{m}} = (x)/(x^3)$ is the unique mxl ideal of R

③ $R = \mathbb{K}[\![x]\!]$ = power series in one variable over $\mathbb{K}$.

Claim: R is a ring.

Operations: • Componentwise addition (degree-by-degree)

$$\Big(\sum_{k\geq 0} a_k x^k\Big) + \Big(\sum_{k\geq 0} b_k x^k\Big) = \sum_{k\geq 0} (a_k+b_k)x^k.$$

• Multiplication: $$\Big(\sum_{k\geq 0} a_k x^k\Big)\Big(\sum_{\ell\geq 0} b_\ell x^\ell\Big) = \sum_{n\geq 0}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)x^n.$$

Q: Why is R local?

Claim: Any $f \in R$ with constant term $\neq 0$ is invertible!

So any $g \in R$ is $g = x^n u$ with $u \in R^\times$ & $n \geq 0$.

Conclusion: $\mathfrak{m} = (x)$ is the unique max'l ideal of R.

Pf of claim WLOG, assume $f = \sum_{n\geq 0} a_n x^n$ with $a_0 = 1$. ($\frac{f}{a_0}$ has this property)

Build f^{-1} one term at a time. Write $g = \sum_{n\geq 0} b_n x^n$ with $fg = 1$

$$\Rightarrow \begin{cases} a_0 b_0 = 1 & \rightsquigarrow b_0 = 1 \\ a_1 b_0 + a_0 b_1 = 0 & \rightsquigarrow b_1 = -a_1 b_0 \\ a_2 b_0 + a_1 b_1 + a_0 b_2 = 0 & \rightsquigarrow b_2 = -a_2 b_0 - a_1 b_1 \end{cases}$$

Assume we have $b_0, \dots, b_{n-1}$. Then

$a_n b_0 + \dots + a_0 b_n = 0$ & $a_0 = 1$

$\Rightarrow b_n = -a_n b_0 - \dots - a_1 b_{n-1} \in \mathbb{K}$.

Obs: In our definition of $+$ & $\cdot$ on $\mathbb{K}[x]$ only finitely many operations in $\mathbb{K}$ were performed to get the coefficient of x^n once n is fixed.

Eg $\sum_{k=0}^{n} a_k b_{n-k}$ & $a_n + b_n$ gave the x^n-coeff of $\cdot$ & $+$.

- The same idea will give us a ring structure on

$$\boxed{\mathbb{K}((x))} = \text{Laurent series} = \left\{ \sum_{j=-N}^{\infty} a_j x^j \quad a_j \in \mathbb{K} \ \forall j \geq -N, \ N \in \mathbb{Z}_{\geq 0} \right\}$$

Fun exercise: This definition of $\cdot$ will not work for the abelian gp

$$\mathbb{K}[[x^{-1}, x]] = \left\{ \sum_{j=-\infty}^{\infty} a_j x^j : a_j \in \mathbb{K} \ \forall j \in \mathbb{Z} \right\}$$

(Because if it did: $\cdots + x^{-2} + x^{-1} + 1 + x + x^2 + \cdots = \frac{-x^{-1}}{1-x^{-1}} + \frac{1}{1-x} = 0$)

Compare coeff of x^k to get $1 = 0$!

Characterizing Local rings

Proposition: R is local if & only if the set of all non-units of R is an ideal of R.

Pf/ ($\Longrightarrow$) Assume $(R, \mathfrak{m})$ is local. Since $\mathfrak{m} \subsetneq R$, we have $\mathfrak{m} \subseteq R \setminus R^\times$.

Conversely if $x \in R \setminus R^\times$, then (x) is a proper ideal & we can find a max'l ideal of R containing x. Since R is local, $x \in \mathfrak{m}$. Thus $R \setminus R^\times \subseteq \mathfrak{m}$.

Therefore, $\mathfrak{m} = R \setminus R^\times$, so $R \setminus R^\times$ is an ideal of $\mathfrak{m}$.

($\Longleftarrow$) If $\mathfrak{m} = R \setminus R^\times$ is an ideal, then $\mathfrak{m}$ is maximal. Any $x \notin \mathfrak{m}$ will be a unit so if $\mathfrak{a} \supsetneq \mathfrak{m}$ is an ideal with $x \in \mathfrak{a} \setminus \mathfrak{m}$, then $\mathfrak{a} = (1) = R$.

Now, if $\mathfrak{b}$ is any proper ideal of R, then $\mathfrak{b} \subseteq R \setminus R^\times$, so $\mathfrak{b} \subseteq \mathfrak{m}$.

Thus, $\mathfrak{m}$ is the unique maximal ideal of R. $\square$

Example: $R = \mathbb{K}[x]/(x^2) = \{ \underbrace{a_0 + a_1\bar{x} + a_2\bar{x}^2}_{=: f(x)} \quad a_0, a_1, a_2 \in \mathbb{K} \}$ is local.

Claim: f is a unit $\Longleftrightarrow$ $a_0 \in \mathbb{K} \setminus \{0\}$ So $R - R^\times = (x)$ is an ideal.

$$(a_0 + a_1\bar{x} + a_2\bar{x}^2)^{-1} = (a_0^{-1} + (-a_1 a_0^{-2})\bar{x} + (a_0^{-1}(-a_0^{-1}a_2 + a_0^{-2}a_1^2))\bar{x}^2)$$

Example 2: Fix $p \in \mathbb{Z}_{\geq 2}$ prime & set

$$R = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; p \nmid b \;\&\; \gcd(a,b)=1 \right\}$$ (usual name $\mathbb{Z}_{(p)}$)

Claim 1: R is a ring (subring of $\mathbb{Q}$)

- $\frac{0}{1}, \frac{1}{1} \in R$ ✓
- $\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1b_2 + a_2b_1}{b_1b_2}$ $\qquad$ $p \nmid b_1,\ p \nmid b_2 \Rightarrow p \nmid b_1 b_2$

  $\Rightarrow p \nmid \frac{b_1b_2}{\gcd(c, b_1b_2)}$ where $c = a_1b_2 + a_2b_1$

  $\Rightarrow \frac{a_1}{b_1} + \frac{a_2}{b_2} \in R$ ✓

- $\frac{a}{b} \in R \Rightarrow \frac{-a}{b} \in R$ ✓
- $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{a_1a_2}{b_1b_2}$ $\qquad$ $p \nmid b_1,\ p \nmid b_2 \Rightarrow p \nmid b_1 b_2$

  $\Rightarrow p \nmid \frac{b_1b_2}{\gcd(c, b_1b_2)}$ where $c = a_1a_2$

  $\Rightarrow \frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \in R$ ✓

Moreover $R^\times = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}_{\neq 0}\ \gcd(a,b)=1\ p \nmid a, p \nmid b \right\}$

$\Rightarrow R^\times \setminus R = (p) = pR$ is an ideal. $\Rightarrow$ R is local.

$\boxed{\text{Nilpotent elements}}$

R = commutative ring

<u>Def</u>: An element $x \in R$ is called <u>nilpotent</u> if $\exists\, n \geq 1$ such that $x^n = 0$

Let $\mathcal{N}$ = set of all nilpotent elements of R.

<u>Lemma</u>: $\mathcal{N}$ is an ideal of R (called <u>nilradical</u>)

<u>Proof</u>: Pick $a, b \in \mathcal{N}$. Then $\exists\, k, \ell \geq 1$ st $a^k = b^\ell = 0$

$$\left.\begin{aligned} \Rightarrow (a \pm b)^{k+\ell} &= \sum_{j=0}^{k+\ell} \binom{k+\ell}{j} a^j b^{k+\ell-j} (\pm 1)^{k+\ell-j} \\ &= \sum_{j=0}^{k} (\pm 1)^{k+\ell-j} \binom{k+\ell}{j} a^j \underbrace{b^{\ell + \overbrace{(k-j)}^{\geq 0}}}_{=0} + \sum_{j=k+1}^{k+\ell} \binom{k+\ell}{j} \underbrace{a^j}_{\substack{=0 \\ (j>k)}} (\pm b)^{k+\ell-j} = 0 \end{aligned}\right\} \Rightarrow a \pm b \in \mathcal{N}$$

Also: $(ra)^k = r a^k = r \cdot 0 = 0$ so $ra \in \mathcal{N}$ $\forall r \in R$ $\square$

⚠ This fails if R is noncommutative.

$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ & $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$E^2 = F^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $E, F \in \mathcal{N}$

But $EF = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin \mathcal{N}$, $(EF)^n = EF$ $\forall n$

$\Rightarrow \mathcal{N}$ neither left nor right ideal of $M_{2\times 2}(\mathbb{C})$.

Ring of Fractions

Motivation 1: Number Theory - Diophantine Equations

$P_1(x_1,\dots,x_m), \dots, P_n(x_1,\dots x_m) \in \mathbb{Z}[x_1,\dots,x_m]$

Q: Find integer solutions to $P_1 = \dots = P_n = 0$.

Approach: Look for solutions over $\mathbb{Q}$, or $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \nmid b\}_{(a:b)=1}$ and "patch the local solutions"

(localization of $\mathbb{Z}$ at (p))

Motivation 2: Geometric viewpoint towards commutative rings.

Commutative ring R = [type] functions on [type] space X with values in $\mathbb{C}$ or other field.

Eg: continuous / polynomial — topological / vector space $\mathbb{K}^n$

Ideals = subsets of functions which vanish on a given subset $Y \subset X$ (closed)

Open sets $\longleftrightarrow$ non vanishing of certain functions [Eg: $GL_2(\mathbb{C}) = (\det \neq 0) \subseteq M_{2\times 2}(\mathbb{C})$]

Localization: Study the behaviour of a space near a point

Def: Fix a commutative ring R & $S \subset R$. We say S is a <u>multiplicatively closed set</u> if:

(i) $0 \notin S$ (ii) $1 \in S$ (iii) $a, b \in S \Rightarrow ab \in S$.

- Next, we define an <u>equivalence relation on $R \times S$</u>:

$$\boxed{(a,s) \sim (b,t) \iff \exists\, s' \in S \text{ with } s'(at - bs) = 0}$$

- Idea comes from $\mathbb{Q}$: $\frac{a}{b} = \frac{c}{d} \iff 1(ad - bc) = 0$ (replace 1 by $s' \in S$).

Pf/ Symmetric (same s') & reflexiveness ($s' = 1$) are clear.

- Transitivity: $(a,s) \sim (b,t)$ & $(b,t) \sim (c,u) \overset{?}{\Rightarrow} (a,s) \sim (c,u)$

From the given relations, $\exists\, s', s'' \in S$ such that

(1) $s'at = s'bs$ & (2) $s''bu = s''ct$

Then: $(s's''t)\,au \underset{(1)\cdot s''u}{=} s's''bsu = s's''sbu \overset{(2)}{=} s'ss''ct$

$= (s's''t)(sc)$

$\Rightarrow (s's''t)(au - sc) = 0$ & $(s's''t) \in S$ since $s', s'', t \in S$ + (iii).

Def The <u>ring of fractions of R relative to S</u>, denoted by $S^{-1}R$ is the set $R\times S/\sim$ with

① Addition: $(a,s)+(b,t)=(at+bs,\, st)$

② Multiplication: $(a,s)\cdot(b,t)=(ab,\, st)$

③ Neutral elements: $0=(0,1)$ & $1=(1,1)$

Standard notation $(a,s)\in S^{-1}R \longleftrightarrow$ "$\frac{a}{s}$". (mimic $\mathbb{Q}$)

Exercise: • Verify that addition and multiplication formulae given above are well-defined (ie independent of class reps.)

• Check that $S^{-1}R$ is a ring with these operations

Note: If $0\in S$, then $S^{-1}R = R\times S/\sim$ is reduced to a single pt since $0(at-bs)=0$ always. $\leadsto S^{-1}R$ would be the "zero ring" which we ignored from the beginning